q_{31} =-0.39270 p_{11}-0.0080261688 p_{31}+0.01210950212 p_{32}+0.67065 p_{33},

q_{32} =-0.39270 p_{12}-0.01014090914 p_{32}-0.6971590909 p_{31}+0.4 p_{33},

q_{33} =-0.39270 p_{13}-0.01460 p_{33}-0.39270p_{31}.

To determine the elements of the matrix P^*, Cramer's method is used to solve a system of linear equations by converting it into a matrix form. The Cramer method uses the determinant of a matrix and another matrix obtained by replacing one of the columns with a vector consisting of the numbers to the right of the equation, namely the elements of the Hermitian matrix. The form of a matrix of order 9x9 as shown below,

$$\begin{pmatrix} -0.00145 & 0.01210 & \dots & 0 \\ -0.69715 & -0.0035 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -0.0146 \end{pmatrix} \begin{pmatrix} p_{11} \\ p_{12} \\ \vdots \\ p_{33} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ -1 \end{pmatrix}$$

Thus, based on Cramer's method, the elements of the matrix P^*,

$$P^* = \begin{pmatrix} p_{11}^* & p_{12}^* & p_{13}^* \\ p_{21}^* & p_{22}^* & p_{23}^* \\ p_{31}^* & p_{32}^* & p_{33}^* \end{pmatrix} = \begin{pmatrix} 1491.420937 & 848.6093281 & -14.45353822 \\ 848.6093281 & 3407.097322 & 1501.987498 \\ -14.45353822 & 1501.987498 & 846.0101797 \end{pmatrix}.$$

Next, matrix P^* is determined whether is negative definite or not by taking into account the eigenvalues. The eigenvalues obtained are $\lambda_1 = 48.1213350185760$, $\lambda_2 = 1386.32710869567$, and $\lambda_3 = 4310.07999498577$. Since all the eigenvalues are positive, then the matrix P^* is positive definite, so the Lyapunov function is obtained as follows,

$$V(x) = f^T(x)P^*f(x).$$

This can be written as

$$\begin{aligned} V(x) = &\left(1043.9946x - 8.6999x^2 - \frac{327.8339yx}{0.004xy+0.08x+0.05y+1} - 383.9979zx - 599.809y + 5.7814z\right) \\ &\left(0.7x - 0.0058x^2 - \frac{0.51yx}{0.004xy+0.08x+0.05y+1} - 0.255zx\right) + \Big(594.0265x - 4.9502x^2 - \\ &\frac{1304.8288yx}{0.004xy+0.08x+0.05y+1} - 166.6114zx - 1784.1731y - 600.7949z\Big)\left(-0.7x + \frac{0.51yx}{0.004xy+0.08x+0.05y+1}\right) \\ &+\Big(-10.1174x + 0.0843x^2 - \frac{773.3849yx}{0.004xy+0.08x+0.05y+1} + 219.4182zx - 712.9871y - \\ &338.4040\Big)(0.255zx + 0.4y - 0.4z). \end{aligned}$$

By substituting the value of the equilibrium point $TE_4 = (1.54525, 0.03933, 2.63987)$ then we get the value $V(x) = 0.2081225355$. Since the value of the equation $V(x) > 0$ then it can be concluded that the equilibrium point TE_4 globally asymptotically stable. If the prey population, immature predator, and mature predator were initially around the interior equilibrium point, then the three populations will tend toward the equilibrium point $TE_4 = (1.54525, 0.03933, 2.63987)$. This means that the three populations will not be extinct for a long period of time.

4. Conclusion

The prey-predator model with Crowly-Martin type functional response and stages structure for predator population has only one non-negative equilibrium point $\boldsymbol{TE_4(x,y,z)} = \left(\boldsymbol{\omega}, \frac{b\sigma\omega+d\sigma\omega-\alpha\omega+b+d}{(b+d)(\sigma\omega+1)\vartheta}, \frac{d(b\sigma\omega+d\sigma-\alpha\omega+b+d)}{\vartheta(\sigma\omega+1)(\theta\omega-c)(b+d)}\right)$. This equilibrium point becomes the only interior equilibrium point and it is locally and globally asymptotically stable when a certain conditions are fulfilled. This means that even if there is a tight interaction or predation on prey over a long period of time, the prey population will still be sustainable, stable, and sustainably maintained. This global stability provides an interpretation that the populations in the ecosystem is under control for a long period of time. As suggestions for further research, the model can be developed by considering some assumptions or consider the time delay and harvesting in the mechanism of growth of each prey and predator population.

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