Studying the Antarctic Glacier Melting Rate through Time Series Decomposition and Smoothing

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Abstract

This STEM paper will study the Time Series Antarctic Glacier Mass from April 2002 to March 2021. The objective of this paper is to forecast the Antarctic Glacier Mass level for 2021 to 2041. Among the four STEM components, the science studied is the geoscience of glaciers, technology included is the GRACE-FO satellites to collect Glacier Ice Sheet Mass data, Engineering focuses on the COVID-19 impact on the Glacier melting rate, and mathematical models like Time Series Decomposition and Smoothing were implemented. Several Time Series Decomposition methods such as the detrended, season-adjusted, and differencing were utilized to detect the strength of time series "Trend" and "Seasonal" components. 12-month seasonal pattern and long-term year to year trend were significantly observed. Although the Glacier meting rate sped up recently before 2020, the COVID-19 situation has slowed down the rate of glacier melting in 2020 in both Antarctic and Greenland. Smoothing models were also utilized to smooth out the random noise component to enhance the forecasting model. Several smoothing models were compared using their forecasting capability regarding both the trend and seasonal components. The prediction interval for forecasting Glacier Mass for 2021 to 2041 would become too wide to predict the future Glacier melting rate for more than 5 years in advance. This STEM Time Series Analysis methodology can be commonly applied to any Time Series data in several fields such as Climatology, Geoscience, Finance, Economics, Production Quality, Real Estate.

Keywords
Time Series, Forecast, Geoscience, Glacier, GRACE-FO

1. Introduction
This project will study the Antarctic Glacier Mass data from 2002-2021 March. The objective is to use the Time Series platform to examine the time series Glacier data to predict the Glacier crisis for the next twenty years (2021-2041).

1.1 Scientific Research Literature and Technology: GRACE-FO
The global climate has been spiraling out of control due to the Global Warming effect (Arnold 2011). The Gravity Recovery and Climate Experiment Follow-On (GRACE-FO) mission is a partnership between NASA and the German Research Centre for Geosciences (GFZ). GRACE-FO aims to test a new technology designed to dramatically improve the already remarkable precision of its measurement system. GRACE-FO is a successor to the original GRACE mission, which orbited Earth from 2002-2017. Global surface mass anomalies are observed by the GRACE-FO satellites. Over land, red colors indicate below-average terrestrial water amounts, while blue colors show above-average water amounts (including ice, snow, soil moisture and groundwater). Over oceans, red colors indicate below-average ocean pressure, while blue colors show above-average pressure. Ocean pressure changes are related to large-scale ocean current variations, as well as overall sea level changes from ocean mass changes.

1.2 Engineering: Antarctic Glacier Melting Crisis

An Antarctic glacier larger than the UK is at risk of breaking up after scientists discovered more warm water flowing underneath it than previously thought. Over the past few years, teams of scientists have been crisscrossing the remote and inaccessible region on Antarctica's western edge to try to understand how fast the ice is melting and what the consequences for the rest of the world might be. "What happens in west Antarctica is of great societal importance," said Dr Robert Larter, a scientist with the British Antarctic Survey and principal investigator with the International Thwaites Glacier Collaboration. Glacier melting is the biggest factor in future sea level rise.

1.4 Mathematics: Time Series and Forecast

Time Series Analysis and Forecasting modeling were utilized on the GRACE-FO Glacier Mass data. Climatology research has used Time Series and Forecasting model such as ARIMA to forecast the weather temperature and study the global warming trend Baillie (2002). In this paper, the Time Series Decomposition and Smoothing Models will be compared on their Forecasting Capabilities for the next 20 years (2021-2041).

2. Data Collection and Sampling Plan

2.1 GRACE-FO Antarctic Glacier Data and Sampling Plan

The data source for this paper is from the NASA GRACE-FO satellites' data of the Antarctic Ice Sheet Mass Trend as shown in Figure 1 which collects monthly averages of the images collected from satellites.

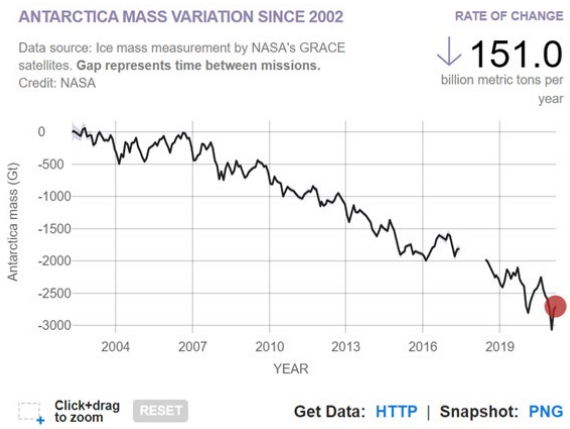


Figure 1. Antarctic Monthly Mass Trend

The Glacier Mass raw data was uploaded to the JMP platform from the NASA GRACE-FO website as shown in Figure 2.

2002
Source

Columns (6/0)
Antarcti...tonnes)
Year
Month
Year-Month
Antarcti...rended)
Antarcti...it cycle)

Rows

All rows	219
Selected	0
Excluded	0
Hidden	0
Labeled	0

	Antarctic mass (Gigatonnes)	Year	Month	Year-Month	Antarctic mass (Gigatonnes) (Detrended)	Antarctic mass (Gigatonnes) (Remove 12 unit cycle)
1	•	2002	1	01/2002	•	•
2	•	2002	2	02/2002	•	•
3	•	2002	3	03/2002	•	•
4	0	2002	4	04/2002	-312.2540246	141.67839272
5	18.36	2002	5	05/2002	-281.1661726	51.559262294
6	•	2002	6	06/2002	•	•
7	•	2002	7	07/2002	•	•
8	-59.82	2002	8	08/2002	-321.1626167	-285.6739764
9	45.54	2002	9	09/2002	-203.0747647	-166.6549122
10	62.69	2002	10	10/2002	-173.1969127	-78.98839272
11	-69.03	2002	11	11/2002	-292.1890607	-102.2292623
12	-49.78	2002	12	12/2002	-260.2112087	34.395583658
13	-48.71	2003	1	01/2003	-246.4133567	130.28564995
14	-200.03	2003	2	02/2003	-385.0055048	25.823976382
15	-171.49	2003	3	03/2003	-343.7376528	40.70491224
16	-43.66	2003	4	04/2003	-203.1798008	98.018392724
17	0.79	2003	5	05/2003	-146.0019488	33.989262294
18	•	2003	6	06/2003	•	•
19	-128.94	2003	7	07/2003	-250.2762448	-307.9356499
20	-122.41	2003	8	08/2003	-231.0183929	-348.2639764
21	-130.92	2003	9	09/2003	-226.8005409	-343.1149122
22	-48.06	2003	10	10/2003	-131.2126889	-189.7383927
23	-107.58	2003	11	11/2003	-178.0048369	-140.7792623
24	-273.11	2003	12	12/2003	-330.8069849	-188.9344163

Figure 2. Glacier Mass Monthly Row Data File

3. Time Series Basic Analysis

Conduct JMP 16 Time Series and Forecasting Platforms on the Glacier Mass data. There are two main objectives in this Section 3: (1) Any COVID-19 Factor on Glacier Mass melting rate in 2020? (2) Can Time Series Analysis detect the "Trend" and "Seasonal" components?

3.1 Study COVID-19 Factor

To visualize Antarctic Glacier Mass trend from 2002-2021, JMP Control Chart Builder platform was used. In Fig.3, the Glacier Mass data was plotted in Individual Control Chart format AIAG (2005), Nelson (1984 & 1985), Wheeler (2004). Y Axis is the Antarctic Mass data and X axis is the Month/Year time domain. Y axis scale was set zero at the 2002 April. The Mass data reported was compared relatively to the 2002 April data in Gigatonnes (GT). The downward trending pattern was observed since 2002 and the downward slope was getting steeper after 2007. An interesting finding is that the Glacier Mass melting rate was slowed down in Antarctic in 2020. This observation may be related to COVID-19 factor. Authors have also found a similar trending observation happened in Greenland in 2020. Between September 2018 and August 2019, the Greenland Ice Sheet set a record for ice loss (532 plus or minus 58 billion metric tons). Between September 2019 and August 2020, the rate of ice loss from the Greenland Ice Sheet was much lower (293 plus or minus 66 billion metric tons), but still above the 2002–2020 average measured by GRACE. Average ice loss for Greenland over the full 18-year record was 268 plus or minus 14 billion metric tons per year. This slow down observation may be due to less Human activity and air pollution globally during COVID-19 pandemic period. Authors have been monitoring this COVID-19 factor and may share more findings in next Antarctic Glacier Mass Time Series publication.

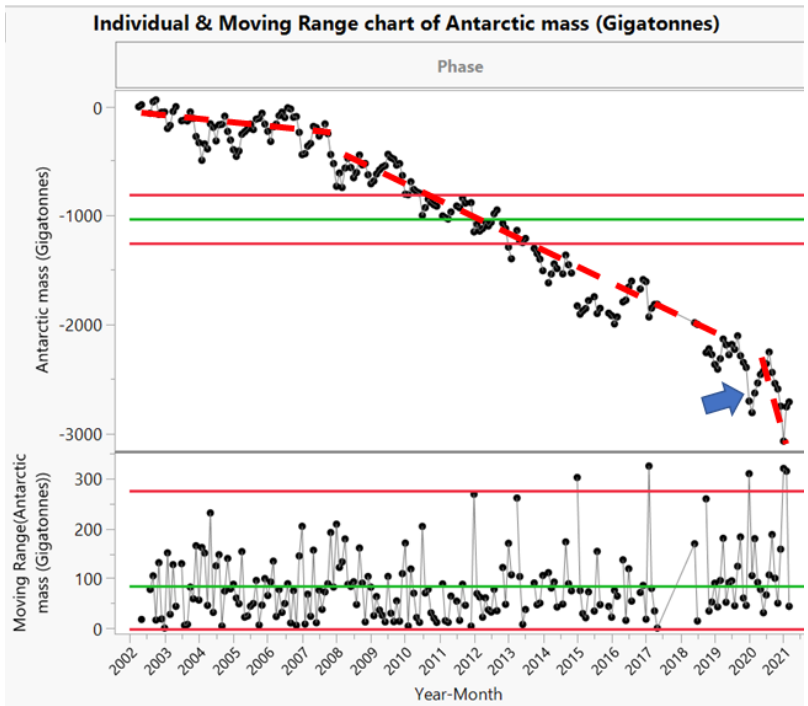
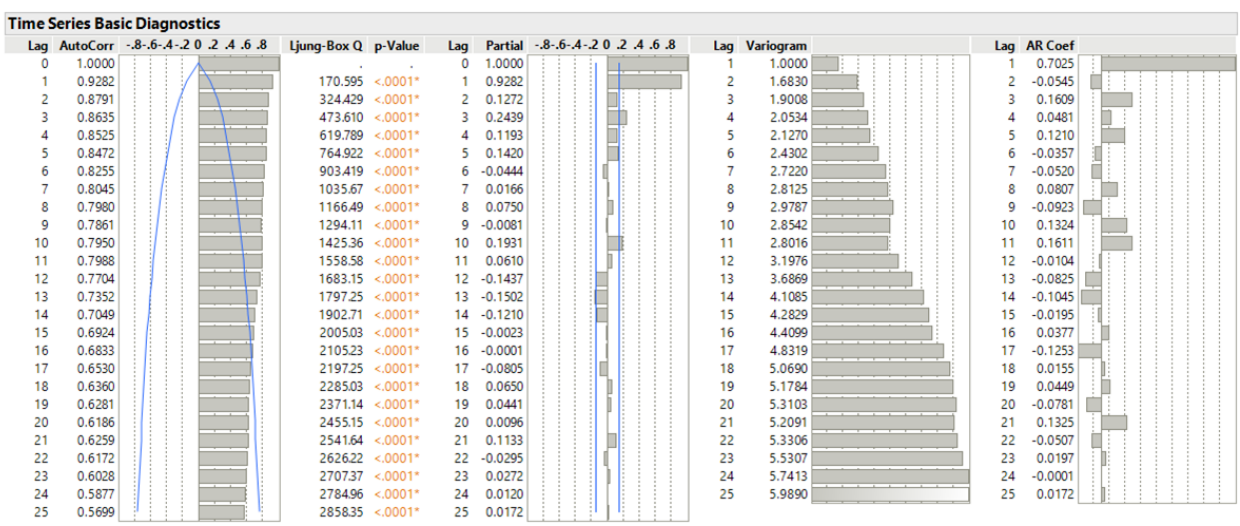


Figure 3. Antarctic JMP Glacier Mass Control Chart Builder Analysis

3.2 Basic Time Series Analysis

Basic Time Series Analysis was conducted as shown in Fig.4. The first focus of the Basic Time Series Analysis is to detect any Seasonal component at any fixed frequency (lag= 12). There was not clear seasonal pattern at lag=12 from Autocorrelation, Partial Correlation, Variogram and AR coefficient in Fig.4. There are two scenario: one is no true seasonal component and the other one is the seasonal component may be masked by the stronger trending component.



Time Series Basic Diagnostics

Lag	AutoCorr	Ljung-Box Q	p-Value
0	1.0000	.	.
1	0.9282	170.595	<.0001*
2	0.8791	324.429	<.0001*
3	0.8635	473.610	<.0001*
4	0.8525	619.789	<.0001*
5	0.8472	764.922	<.0001*
6	0.8255	903.419	<.0001*
7	0.8045	1035.67	<.0001*
8	0.7980	1166.49	<.0001*
9	0.7861	1294.11	<.0001*
10	0.7950	1425.36	<.0001*
11	0.7988	1558.58	<.0001*
12	0.7704	1683.15	<.0001*
13	0.7352	1797.25	<.0001*
14	0.7049	1902.71	<.0001*
15	0.6924	2005.03	<.0001*
16	0.6833	2105.23	<.0001*
17	0.6530	2197.25	<.0001*
18	0.6360	2285.03	<.0001*
19	0.6281	2371.14	<.0001*
20	0.6186	2455.15	<.0001*
21	0.6259	2541.64	<.0001*
22	0.6172	2626.22	<.0001*
23	0.6028	2707.37	<.0001*
24	0.5877	2784.96	<.0001*
25	0.5699	2858.35	<.0001*

Lag	Partial
0	1.0000
1	0.9282
2	0.1272
3	0.2439
4	0.1193
5	0.1420
6	-0.0444
7	0.0166
8	0.0750
9	-0.0081
10	0.1931
11	0.0610
12	-0.1437
13	-0.1502
14	-0.1210
15	-0.0023
16	-0.0001
17	-0.0805
18	0.0650
19	0.0441
20	0.0096
21	0.1133
22	-0.0295
23	0.0272
24	0.0120
25	0.0172

Lag	Variogram
1	1.0000
2	1.6830
3	1.9008
4	2.0534
5	2.1270
6	2.4302
7	2.7220
8	2.8125
9	2.9787
10	2.8542
11	2.8016
12	3.1976
13	3.6869
14	4.1085
15	4.2829
16	4.4099
17	4.8319
18	5.0690
19	5.1784
20	5.3103
21	5.2091
22	5.3306
23	5.5307
24	5.7413
25	5.9890

Lag	AR Coef
1	0.7025
2	-0.0545
3	0.1609
4	0.0481
5	0.1210
6	-0.0357
7	-0.0520
8	0.0807
9	-0.0923
10	0.1324
11	0.1611
12	-0.0104
13	-0.0825
14	-0.1045
15	-0.0195
16	0.0377
17	-0.1253
18	0.0155
19	0.0449
20	-0.0781
21	0.1325
22	-0.0507
23	0.0197
24	-0.0001
25	0.0172

Figure 4. Basic Time Series Analysis on Antarctic Glacier Mass Data

From the Fig.3 Control Chart, the downward trending pattern was significant and the Glacier melting pattern should be highly depended on the Seasonal component (12 months/year). Faster melting during the hot months and slower melting during the cold months. Therefore, the second scenario was more likely to explain the Fig.4 result that the

seasonal component was masked by the stronger trend component. What basic Time Series Analysis could help confirm the second scenario? Spectral Density plot can detect the lag pattern. Spectral Density peaked at lag ~ 12 (Seasonal) as shown in Fig.5 (left period plot). The right frequency plot is the reverse lag plot of the Fig.5. Both plots would detect the identical seasonal lag =12. Spectral Density is more powerful to detect the seasonal lag against strong trend component.

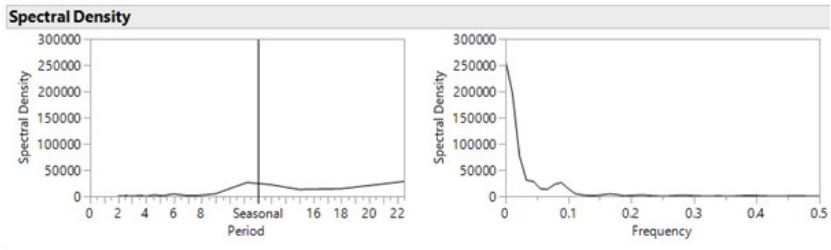


Figure 5. Time Series Spectral Density

4 Time Series Decomposition Analysis

In Section 4, two time series decomposition methods would be utilized to analyze the Antarctic Glacier Mass data: (1) Detrend, and (2) Season-Adjusted.

4.1 Time Series Decomposition- Detrend

Time series data can be decomposed to several components such as “Trend”, “Seasonal” and Cyclic”, “Random” Box (2006), Hyndman (2018). Trend Component: a trend exists when there is a long-term increase or decrease in the data. It does not have to be linear. Sometimes we will refer to a trend as “changing direction,” when it might go from an increasing trend to a decreasing trend. A seasonal pattern occurs when a time series is affected by seasonal factors such as the time of the year or the day of the week. Seasonality is always of a fixed and known frequency. Cyclic: A cycle occurs when the data exhibit rises and falls that are not of a fixed frequency. In general, the average length of cycles is longer than the length of a seasonal pattern, and the magnitudes of cycles tend to be more variable than the magnitudes of seasonal patterns. To further detect the Seasonal component strength, the trend component should be removed by the “detrending” method. JMP Time Series “Detrend” decomposition was conducted as shown in Fig.6. The linear trend model Beta1 is the slope of the Mass downward trend. After detrended, more like random noise pattern. Though, observe certain non-linear pattern in the Time Series Plot which may indicate the trend pattern is not linear.

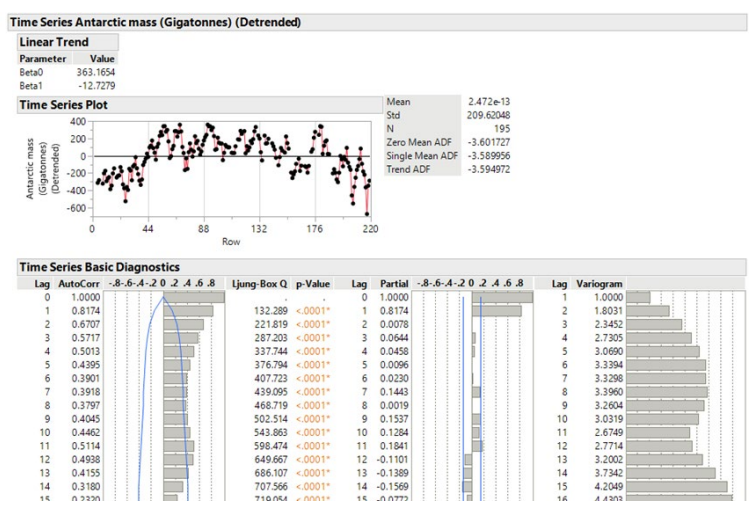


Figure 6. Time Series “Detrend” Decomposition Analysis

In Fig.6 bottom Time Series Plot, as compared to Fig. 4 before "detrended", stronger Autocorrelation and Variogram at lag= 12 were observed. After detrended the Linear model, the seasonal component pattern is clear. Autocorrelation values and ACF plot have been displayed on the left portion. ACF plot has detected seasonal lags at 12. This could be interpreted that each the seasonal "month" factor is "12" months within each year for positive Autocorrelation. The Variogram plot on the right can provide similar information by calculating the standard deviation at every lag level for the entire data distribution. Lag 12 has shown the lowest standard deviation in the Variogram plot which means the dispersion among the temp data every 12 months showing the lowest variance level. Interestingly, the Variogram plot has shown the largest peak at lag 6 which is related to the Glacier melting speed faster in hot months and slower in cold months. The Autocorrelation and Variogram plots have clearly described the Seasonal pattern.

4.2 Time Series Decomposition- Season Adjusted

Another powerful Time Series Decomposition is to remove the Seasonal Component (season-adjusted, Shiskin, (1967) as shown in Fig.7. Once seasonal lag =12 was identified in previous Autocorrelation analysis, another JMP function can remove the seasonal component by searching the optimal Cosine Transfer function. JMP platform would remove the seasonal component and conduct the Time Series Analysis in the bottom chart. The new transformed data distribution has shown a strong downward seasonal pattern. This Cosine transformation has further indicated that the seasonal lag is at 12 only. For the Antarctic Glacier Mass study, 12 months seasonal lag is very obvious. For other more complicated data, using both AutoCorrelation/Variogram and Remove Seasonal Component methods may be necessary to validate the seasonal lag which is critical for the Time Series Forecasting part in later Section 5. Sometimes, the data may have double seasonal components and may need to use "Twice- Remove Seasonal Component" method (Seasonal ARIMA) which is not covered in this paper.

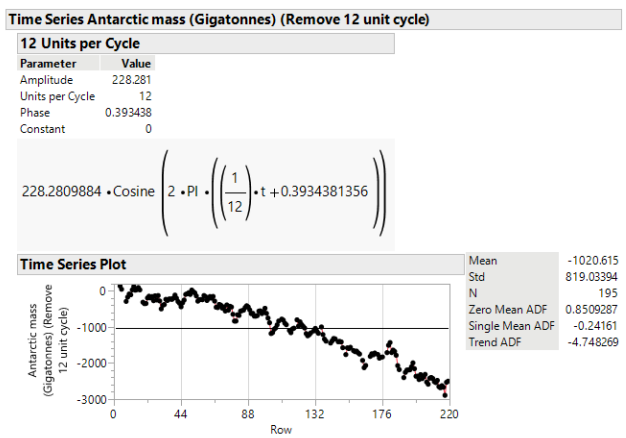


Figure.7 Remove Seasonal Component Analysis

After removed the Seasonal component, the new time series data has shown weaker Spectral Density and smoother Trend in Time Series Plot in Fig.8.

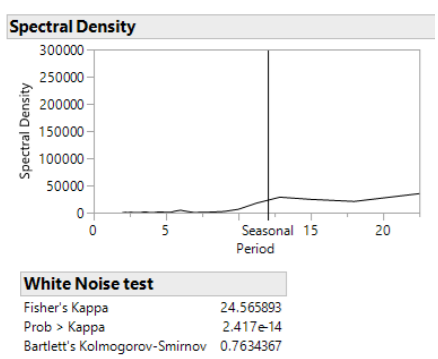


Figure. 8 Spectral Density Plot after removed the Seasonal Component

5 Time Series Smoothing Method

Smoothing technique in Time Series is used to remove the random noise component in order to detect the long term trend component for enhancing the forecasting capability. In Section 5, we would study and compare different Time Series Smoothing techniques.

5.1 Simple Moving Average Smoothing

The simplest smoothing technique is Simple Moving Average Smoothing. The smoothing formula is to use the "un-weighted" moving average over the smoothing width period. Optimizing the Smoothing Width is an important subject for Time Series Analysis. If the width too small, the smoothed curve may be still too noisy to detect the seasonal or trend components

If the width too large, the smoothed curve may also remove the Seasonal component. Double Smoothed method with small smoothing width is better for Smoothing with Seasonal Component. As shown in Fig.9, single and double moving average smoothed methods are compared. The smoothing width was chosen optimally at 5 for both single and double methods. Since the seasonal component at lag=12 is very strong, the simple moving average smoothing methods may not be necessary to smooth out the original Glacier data.

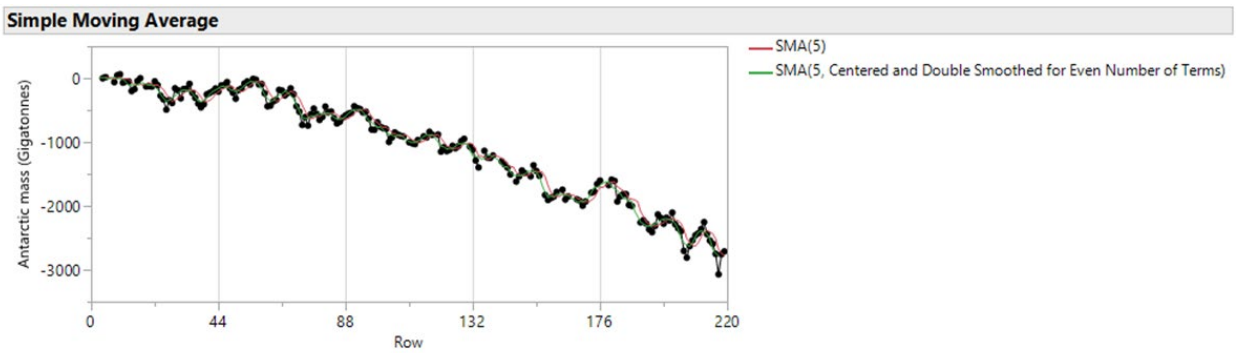


Figure 9. Simple Moving Average Smoothing Analysis

5.2 Exponential Smoothing

Exponential smoothing has proposed in the late 1950s [Brown,1959; Holt; 1957; Winters, 1960}, and has motivated some of the most successful time series forecasting methods. Unlike previous Simple Moving Average smoothing, forecast produced using exponential smoothing methods are weighted averages of pass observations, with the weights decaying exponentially as the observations get older. The more recent the observation the higher the associated weight. This framework generates reliable forecasts quickly and for a wide range of time series, which is a great advantage and of major importance to applications in industry.

Before showing the JMP Time Series Exponential Smoothing analysis, several Exponential Smoothing techniques are introduced below:

- Simple Exponential Smoothing: suifig for forecasting data with no clear trend or no seasonal pattern
- Double (Brown) Exponential Smoothing: much stronger smoothing power and can reduce the residual amplitude. The potential risk is its forecasting capability with wider Prediction Interval
- Holt's Linear Exponential Smoothing: the forecast function is trending but no seasonal pattern. The h-step-ahead forecast is equal to the last estimated level plus h times the last estimated trend value.
- Damped Trend Linear Exponential Smoothing: damps the Holt's Linear Exponential Smoothing. Useful for longer forecasting horizons.
- Seasonal Exponential Smoothing: sustain the trend Components of the last few data points to carry the Seasonal component in Forecasting but missing the trend component.
- Holt-Winters' Seasonal Smoothing: extends the Holt's Linear trend smoothing to keep the Seasonality pattern.

Different Time Series Data may fit to different Exponential Smoothing Model. In Fig.10, JMP Exponential Smoothing platforms were conducted and compared. Models are ranked and sorted based on the AIC (default setting). Can consider other selection criteria: Variance, BIC Burnham (2004 & 2011), RSquare, Likelihood, MAPE and MAE. Top two exponential smoothing models are " Seasonal and Winters (additive). Further study would be addressed later in 5.3 Forecast section.

Model Comparison

Report	Graph	Model	DF	Variance	AIC ^	SBC	RSquare	-2LogLH	Weights	.2 .4 .6 .8	MAPE	MAE
☑	☑	Seasonal Exponential Smoothing(12, Zero to One)	142	9988.5582	1745.5400	1751.4797	0.988	1741.54	0.731059		37.955385	58.871481
☑	☑	Winters Method (Additive)	141	10059.399	1747.5400	1756.4495	0.988	1741.54	0.268941		37.955385	58.871481
☑	☑	Linear (Holt) Exponential Smoothing	159	15372.821	2011.9380	2018.1008	0.981	2007.938	0.000000		47.898223	81.114595
☑	☑	Double (Brown) Exponential Smoothing	160	16145.228	2018.4599	2021.5413	0.980	2016.4599	0.000000		48.273644	82.019106
☑	☑	Damped-Trend Linear Exponential Smoothing	175	11019.312	2164.8346	2174.3799	0.985	2158.8346	0.000000		54.371713	74.816753

Figure 10. Exponential Smoothing study on Antarctic Glacier Mass Data

The two main benefit for using the smoothing technique is to decompose the time series components and to forecast the future points reliably. For this Glacier Mass study, the forecasting should carry both 12-months seasonal component, and the long-term downward trending. As shown in Fig.11, some exponential smoothing method may carry both seasonal and trend components well in Forecasting but some may miss one or both components. The bottom residual ACF and PACF plots may indicate how effective of the smoothing technique to sustain the Seasonal component at lag=12.

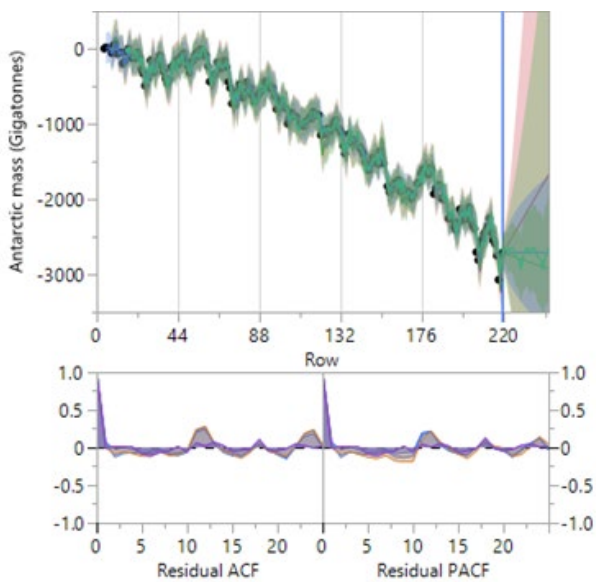


Figure 11. Exponential Smoothing Analysis on the Antarctic Glacier Mass data.

5.3 Time Series Forecast based on Exponential Smoothing Methods

To predict the future points (point estimation of the mean and prediction interval of the error), we need to utilize the previous decomposition algorithm. The forecasting formula should include the long term trend component, carry the seasonal component momentum, and the uncertain error level. Time series analysis has used the decomposition technique to quantify the seasonal and trend components. Use the smoothing technique to smooth out the noise to enhance the signal of "Seasonal" and "Trend" components. Time series has also used the more adv. smoothing techniques such as "moving average", "exponential smoothing" and "state space smoothing" Hyndman (2008) to estimate the error term for forecasting purpose.

In Fig.12, JMP Time Series Forecast analysis was conducted based on the two Exponential Smoothing methods: Holt Linear Trend smoothing and Seasonal smoothing methods. Seasonal lag=12 has been assigned in Forecasting analysis.

The forecasting analysis has duplicated the seasonal pattern. The linear trend was significant in Holt smoothing method. In general, Seasonal smoothing method has observed a better goodness of fit as shown on most criteria on the model summary. The only concern of the Seasonal smoothing is "Invertible". This paper won't address this "invertibility" subject. Authors may address this "invertibility" in the Time Series ARIMA model since "Invertibility" is very critical to the "MA- Moving Average" component on smoothing out the error terms for Forecasting.

Model: Linear (Holt) Exponential Smoothing

Model Summary

DF	159	Stable	Yes
Sum of Squared Innovations	2444278.53	Invertible	Yes
Sum of Squared Residuals	2455646.88		
Variance Estimate	15372.821		
Standard Deviation	123.987181		
Akaike's 'A' Information Criterion	2011.93803		
Schwarz's Bayesian Criterion	2018.10084		
RSquare	0.98122362		
RSquare Adj	0.98112532		
MAPE	47.8982231		
MAE	81.1145954		
-2LogLikelihood	2007.93803		

Parameter Estimates

Term	Estimate	Std Error	t Ratio	Prob>|t|
Level Smoothing Weight	1.0000000	0.0849375	11.77	<.0001*
Trend Smoothing Weight	0.2131561	0.0804985	2.65	0.0089*

Model: Seasonal Exponential Smoothing(12, Zero to One)

Model Summary

DF	142	Stable	Yes
Sum of Squared Innovations	1418375.27	Invertible	No
Sum of Squared Residuals	1486971.64		
Variance Estimate	9988.55823		
Standard Deviation	99.9427748		
Akaike's 'A' Information Criterion	1745.54004		
Schwarz's Bayesian Criterion	1751.47967		
RSquare	0.98777874		
RSquare Adj	0.98771159		
MAPE	37.9553854		
MAE	58.8714813		
-2LogLikelihood	1741.54004		

Figure 12. Compare Holt Linear Vs. Seasonal smoothing methods

After compared the goodness fit of two Exponential Smoothing methods, the Forecasting capability would be further compared in Fig.13. The left plot is for the Holt's Linear method and the right plot is for the Seasonal method. The Linear method has observed the linear trend component in the Forecasting with much wider prediction interval. The Seasonal method has observed the seasonal component in the Forecasting with narrower prediction interval. There are two challenges observed here: (1) is any Time Series Model can forecast the future points based on both the Trend Component and the Seasonal Component, (2) it would be a concern if the prediction interval is too wide at longer horizontal time from today. To address these two concerns, more powerful Time Series Forecast methods may be necessary. Authors would share the ARIMA and State Space smoothing methods in next publication.

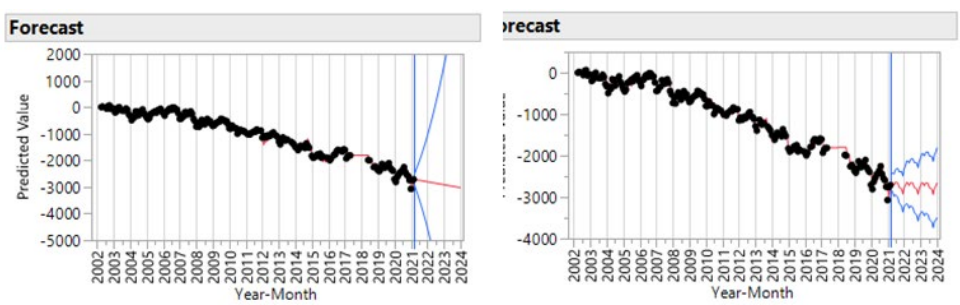


Figure 13. Time Series Forecasting of Antarctic Glacier Mass data.

5 Conclusions

The STEM approach is adopted on analyzing the Time Series Glacier Mass data file from 2002-2021. Earth science and geoscience regarding the glacier melting crisis were learned, and the impact of COVID-19 on the rate of melting was also studied. The Time Series Analysis can decompose the Seasonal and Trend Components and smooth out the Error Component for enhancing the Forecasting capability for 2021 to 2041.

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