Multilevel Reorder Strategy-based Supply Chain Model

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Abstract

We investigate the stochastic integrated inventory model wherein the buyer's lead time demand follows the mixture of normal distributions. Due to the high acquisition cost of land, we assume that buyer's maximum permissible storage space is limited and therefore adds a space constraint to the respective inventory system. Besides, it is assumed that the manufacturing process is imperfect and produces defective units, and hence each lot received by the buyer contains percentage defectives. The paper also considers controllable lead time components and ordering cost for the system. Based on lead-time components, a multilevel reorder strategy-based supply chain model is developed for the proposed system, and a Lagrange multiplier method is applied to solve the problem to reduce the expected inventory cost of both buyer and vendor. We develop a solution procedure to find the optimal values and show the applicability of the model and solution procedure in numerical examples.

Keywords
Integrated vendor-buyer model, Imperfect production, Stochastic lead time, Nonlinear constrained optimization, Mixture of normal distributions

1. Introduction
The integrated inventory model of both buyer and vendor has been received a lot of attention in the past decade. Researchers have proposed that having better coordination of all parties involved in a supply chain will lead to benefit the entire supply network rather than a single company. (Goyal, 1977) and (Banerjee, 1986) were the first researchers that aim to obtain coordinated inventory replenishment decisions. In the mentioned studies, demand and lead time were assumed to be deterministic. However, demand or lead time across different industries is distributed stochastically, so it is relevant and meaningful to consider uncertainty in integrated inventory models. Also, with the successful Japanese experience of using Just-In-Time (JIT) production, the benefits associated with controlling the lead time can be perceived. These benefits include lower safety stock, improve customer service level, and, thus, increase the competitiveness in the industry. To address this issue, researchers started to extend the previously established models by developing lead time reduction inventory models under various crashing cost functions. (Liao & Shyu, 1991) were the first researchers to introduce variable lead times in the inventory model. In their model, they assumed that to reduce the lead time to a specified minimum duration, lead time could be decomposed into several components with different crashing costs. Since then, many researchers have made significant contributions to controllable lead-time literature. ((Ouyang et al., 2004), (Tahami et al., 2019)).

Ordering cost reduction has become an essential aspect of business success and has recently attracted considerable research attention. It can be shown that ordering cost control can affect directly or indirectly the ordering size, service level, and business competitiveness. Integrated vendor-buyer inventory models were typically developed to consider fixed ordering costs. However, in some practical situations, the cost of ordering can be controlled and reduced in various ways. It can be achieved through workforce training, process changes, and special equipment acquisition. (Porteus, 1985) was the first researcher proposed an inventory model considering an investment in reducing set up cost. (Chang et al., 2006) suggested that in addition to controllable lead time, ordering cost could also be considered as a controllable variable. They proposed that the buyer ordering cost could be reduced by the additional crashing cost, which could be defined as a function of lead time length and ordering lot size. Later, other researchers developed setup/order cost reduction inventory models under various assumptions. ((Lou & Wang, 2013), (Tahami et al., 2016))
Assuming that buyer possesses infinite storage capacity is not realistic. In contrast, most probably, the buyer storage capacity is limited. Most previous research on space-constrained inventory problems focused on deterministic demand

having multiple items in the system (Haksever & Moussourakis, 2005), and few papers considered stochastic demand models. One of the first authors proposed inventory models with storage space constraint in the stochastic environment was (Veinott, 1965). (Hariga, 2010) proposed a single item stochastic inventory model with a random demand wherein buyer storage space was limited. In the proposed model, the order quantity and reorder point were considered as decision variables. (Moon & Ha, 2012) presented a multi-item EOQ model with limited storage space and random yields. They solved the model using the Lagrange multiplier method. In our proposed model, we employ the same Lagrange multiplier technique to solve the non-linear objective function.

One of the assumptions of the inventory management literature is that the quality of the product in a lot is perfect. In practice, however, a received lot may contain some defective items. If there is a possibility that a lot contains defective items, the firm may issue a larger order than was originally planned to guarantee the satisfaction of customer demand. (Huang, 2002) proposed an integrated inventory policy for a vendor-buyer model wherein manufacturing process was imperfect and a lot transferred to the buyer contained a fixed fraction of defectives with known probability distribution function. In the previously discussed research, demand was assumed deterministic. However, demand is stochastically distributed in its nature in most industries ((Fazeli et al., 2020), (Yahoodik et al., 2020)). Very few papers have been published for stochastic demand integrated vendor buyer inventory models under defective items considerations. (Ho, 2009) investigated an integrated vendor-buyer inventory system with defective goods in the buyer's arrival order lot. Utilizing the result of a basic theorem from renewal reward processes ((Ross, 1996)) and minimax distribution free procedure for unidentified lead time demand distribution, she obtained the minimum total expected annual cost.

In the case of probabilistic demand, as can be seen in various industries, it is prevalent that the demands by the different customers are not identical, and the distribution of demand for each customer can be adequately approximated by a distribution. The overall distribution of demand is then mixture. So, we cannot use only a single distribution. (Lee et al., 2004) proposed a one-sided inventory system with defective goods wherein the lead time demand followed a mixture of normal distributions and found buyer's optimal inventory strategy when reorder point, lead time, and ordering quantity were the decision variables. In the previously mentioned research, one facility (e.g., a buyer) is assumed to minimize its own cost. This one-sided-optimal- strategy is not appropriate for the global market. Therefore, in this study, we consider a mixture of normally distributed lead time demand for integrated single-vendor single-buyer inventory model rather one facility. Also, the present paper extends the mentioned works for a multi-reorder level inventory system based on lead time components. Besides, we assume that the lead time is controllable and transportation and setup times and their crashing cost act independently. Also, in order to fit some real environment, transportation time crashing cost is presented as a function of reduced transportation time and the quantities in the orders. In this paper, a random space constraint for random demand and positive lead time is considered when maximum permissible storage space is restricted. Also, the manufacturing process is considered imperfect, and defective items are found in the buyer inspection process. The objective is to minimize joint inventory expected cost by simultaneously optimizing ordering quantity, reorder points of different batches, ordering cost, setup time, transportation time, production time and a number of deliveries under imperfect production process and buyer space constraint while the lead time demand follows a mixture of normal distributions. The Lagrangian method is applied to solve the problem.

The rest of the paper is organized as follows. In section 2, the notations and assumptions are given. In section 3, we present the mathematical model. In section 4, a numerical example and sensitivity analysis are provided to illustrate the model and its solution procedure. Finally, we conclude the paper.

2. Notations and assumptions

2.1 Notations

Following notations have been used through the paper:

Q Buyer's order quantity, as a decision variable

r Buyer's Reorder point, as a decision variable

A Buyer's ordering cost at the time zero, as a decision variable

t Transportation time, as a decision variable

s Setup time, as a decision variable

m The number of lots in which the product is delivered from the vendor to the buyer in one Production cycle, a positive integer, as a decision variable

γ Defective rate in an order lot, $\gamma \epsilon [0,1)$ and is a random variable

$g(\gamma)$ Probability density function (p.d.f.) of γ
D Annual demand for buyer
P Production rate in units per unit time
p $1/P$
a Vendor's setup cost per set up at the time zero
π Buyer's stock out cost per unit at the time zero
h_v Vendor's holding cost per item per year at the time zero
h_{b1} Buyer's holding cost per non-defective item per unit time
h_{b2} Buyer's holding cost per defective item per year at the time zero
λ Screening rate
S^c Unit screening cost
W^c Vendor's unit treatment cost (include warranty cost) of defective items
f Space used per unit
F Maximum permissible storage space
$I(A)$ Buyer's capital investment required to achieve ordering cost A, $0 < A \leq A_0$
b Percentage decrease in ordering cost A per dollar increase in investment $I(A)$
θ Fractional opportunity cost of capital investment per year
C_{pu} Buyer's purchasing cost per unit at the time zero
c_s Vendor's Setup cost per setup at the time zero
c_{pr} Vendor's Production cost per unit at the time zero
A_0 Buyer's original ordering cost per order
X Demand during lead time, as a random variable
X^+ Maximum value of x and 0
$E(\cdot)$ Mathematical expectation

2.2 Assumptions

1. A single-vendor and single-buyer for a single product are considered in this paper.
2. The vendor's production rate for the perfect items is finite and greater than the buyer's demand rate, i.e., $P(1 - M_\gamma) > D$, where M_γ, P and D are given.
3. The buyer orders a lot of size mQ, and the vendor manufactures a lot of size mQ, but transfer a shipment of size Q to the buyer. Once the vendor produces the first Q units, he will deliver them to the buyer. After the first delivery and buyer's inspection, the vendor will schedule successive deliveries every $\frac{Q(1-M_\gamma)}{D}$ units of time.
4. An arrival lot, Q, may contain some defective goods and the proportion defective, γ, is a random variable which has a PDF, $f(\gamma)$, with finite mean M_γ and variance V_γ. Upon the arrival of an order, all the items are inspected (the screen rate is λ) by the buyer, and defective items in each lot will be discovered and returned to the vendor at the time of delivery of the next lot. Hence, the buyer will have two extra costs: inspection cost and defective items holding cost.
5. During the screening period, the on-hand non-defective inventory is larger or equal to the demand.
6. We assume that the capital investment, $I(A)$, in reducing buyer's ordering cost is a logarithmic function of the ordering cost A. That is,

$$I(A) = \frac{1}{\delta} ln\left(\frac{A_0}{A}\right) \quad for \;\; 0 < A \leq A_0$$

Where δ is the fraction of the reduction in A per dollar increase in investment.
7. Setup time s consists of n^s mutually independent components. The ith component has a normal duration NS_i and minimum duration MS_i, $i = 1,2, \dots, n^s$. If we let $s_0 = \sum_{j=1}^{n^s} NS_j$ and s_i be the length of setup time with components 1,2,…,i, crashed to their minimum duration, then s_i can be expressed as $s_i = \sum_{j=1}^{n} NS_j - \sum_{j=1}^{i}(NS_j - MS_j)$, $i = 1,2, \dots, n^s$ and the setup time crashing cost is given by $CS(s) = c_{si}(s_{i-1} - s) + \sum_{j=1}^{i-1} c_{sj}(NS_j - MS_j)$.
8. The transportation time t consists of n^t mutually independent components. The ith component has a normal duration NT_i and minimum duration MT_i, $i = 1,2, \dots, n^t$.

9. For the ith component of transportation time, the crashing cost per unit time c_{ti}, depends on the ordering lot size Q and is described by $c_{ti} = a_i + b_i Q$, where $a_i > 0$ is the fixed cost, and $b_i > 0$ is the unit variable cost, for $i = 1,2, \dots, n^t$.
10. For any two crash cost lines $c_{ti} = a_i + b_i Q$ and $c_{tj} = a_j + b_j Q$, where $a_i > a_j$, $b_i < b_j$, for $i \neq j$ and $i, j = 1,2, \dots, n^t$, there is an intersection point Q^S such that $c_{ti} = c_{tj}$. These intersection points are arranged in ascending order so that $Q_0^S < Q_1^S < \cdots < Q_w^S < Q_{w+1}^S$, where $Q_0^S = 0, Q_{w+1}^S = \infty$ and $w \leq n^t(n^t - 1)/2$. For any order quantity range (Q_i^S, Q_{i+1}^S), $c_i s$ are arranged such that $c_1 \leq c_2 \leq \cdots \leq c_{n^t}$, and the lead time components are crashed one at a time starting with the component of least c_i, and so on.
11. Let $t_0 \equiv \sum_{j=1}^{n^t} NT_j$ and t_i be the length of transportation time with components $1,2, \dots, i$ crashed to their minimum duration, then t_i can be expressed as $t_i = t_0 - \sum_{j=1}^{i} (NT_j - MT_j)$, $i = 1,2, \dots, n^t$ and the transportation time crashing cost per cycle $CT(t)$ is given by $CT(t) = c_{ti}(t_{i-1} - t) + \sum_{j=1}^{i-1} c_j (NT_j - MT_j)$, where $t \epsilon [t_i, t_{i-1}]$, and $c_j = a_j + b_j Q$ for $j = 1,2, \dots, i$.
12. We consider the deterministic lead time L and assume that the demand for the lead time X follows the mixture of normal distributions, $F_* = \alpha F_1 + (1 - \alpha) F_2$, where F_1 has a normal distribution with finite mean μ_1 and standard deviation $\sigma\sqrt{L}$ and F_2 has a normal distribution with finite mean μ_2 and standard deviation $\sigma\sqrt{L}$. Therefore, the lead time demand, X, has a mixture of probability density function (PDF) which is given by

$$f(x) = \alpha \frac{1}{\sqrt{2\pi}\sigma\sqrt{L}} \times exp\left[-\frac{1}{2}\left(\frac{x - \mu_1 L}{\sigma\sqrt{L}}\right)^2\right] + (1 - \alpha)\frac{1}{\sqrt{2\pi}\sigma\sqrt{L}} \times exp\left[-\frac{1}{2}\left(\frac{x - \mu_2 L}{\sigma\sqrt{L}}\right)^2\right]$$

Where $\mu_1 - \mu_2 = k_1 \sigma/\sqrt{L}$ or $\mu_1 L - \mu_2 L = k_1 \sigma\sqrt{L}$, $k_1 > 0$, $-\infty < x < \infty$, $0 \leq \alpha \leq 1$, $\sigma > 0$. Moreover, the mixture of normal distributions is unimodal for all α if $(\mu_1 - \mu_2)^2 < 27\sigma^2/8L$ or $k_1 < \sqrt{\frac{27}{8}}$. Also, when $(\mu_1 - \mu_2)^2 > 4\sigma^2/L$ or $k_1 > 2$, at least we can find a value of α ($0 \leq \alpha \leq 1$), which makes the mixture of normal distributions to be a bimodal distribution.
13. The reorder point r = expected demand during lead time + safety stock (ss), and $ss = k \times$ (standard deviation of lead time demand), that is $r = \mu_* L + k\sigma_*\sqrt{L}$, where $\mu_* = \alpha\mu_1 + (1 - \alpha)\mu_2$, $\sigma_* = \sqrt{1 + \alpha(1 - \alpha)k_1^2}\sigma$, $\mu_1 = \mu_* + (1 - \alpha)k_1\sigma/\sqrt{L}$, $\mu_2 = \mu_* - \alpha k_1 \sigma/\sqrt{L}$, and k is the safety factor which satisfies $P(X > r) = 1 - p\Phi(r_1) - (1 - p)\Phi(r_2) = q$, where Φ represents the cumulative distribution function of the standard normal random variable, q represents the allowable stock-out probability during L, $r_1 = (r - \mu_1 L/\sigma\sqrt{L}) = (r - \mu_* L/\sigma\sqrt{L}) - (1 - \alpha)k_1$, and $r_2 = (r - \mu_2 L/\sigma\sqrt{L}) = (r - \mu_* L/\sigma\sqrt{L}) + k_1\alpha$.
14. Lead time for the first shipment is proportional to the lot size produced by the vendor and consists of the sum of setup, transportation and production time, i.e., $L(Q, s, t) = s + pQ + t$. For shipments 2,…,m only transportation time has to be considered for calculating lead time, i.e., $L(t) = t$. Since, due to $P > D$, shipments 2,…,m have been completed when the order of buyers arrives. Hence, considering a mixture of normal distributions, the lead time demand for the first batch, X^1, has a probability density function $f\left(x^1, \mu_1 L(Q, s, t), \mu_2 L(Q, s, t), \sigma\sqrt{L(Q, s, t)}, \alpha\right)$ with means $\mu_1 L(Q, s, t), \mu_2 L(Q, s, t)$ and standard deviation $\sigma\sqrt{L(Q, s, t)}$ and for the other batches, the lead time demand, X^2, has a probability density function $f\left(x^2, \mu_1 L(t), \mu_2 L(t), \sigma\sqrt{L(t)}, \alpha\right)$ with means $\mu_1 L(t), \mu_2 L(t)$ and standard deviation $\sigma\sqrt{L(t)}$.

3. Model formulation

As mentioned in assumption 3, the buyer orders mQ units, and the vendor delivers the order quantity of size Q to the buyer in m batches. As stated in assumption 4, each received lot contains a defective percentage, γ, of defective items which is a probabilistic variable with finite mean M_γ and variance V_γ. Hence, the expected number of non-defective items in each shipment is $(1 - M_\gamma)Q$. Hence, considering m shipment, the expected cycle length for vendor and buyer is $mQ(1 - M_\gamma)/D$ and the buyer order quantity is mQ. Before the product is sold to end customers, all the received items are inspected by the buyer at a fixed screening rate λ. Hence, the duration of the screening period of the buyer in each shipment is Q/λ. The length between shipments, $Q(1 - M_\gamma)/D$, are longer than the screening period.

3.1 Buyer's total expected cost per unit time

Each arriving lot contains a percentage of defective items. In each of the successive m shipments, the number of non-defective items is $(1 - \gamma)Q$ and the length of the shipping cycle is $(1 - \gamma)Q/D$. When the inventory of each item reaches to reorder level, management places an order of amount Q. Due to random demand, the shortage may occur at the buyer side. The expected shortage for the first batch is equal to $E(X^1 - r^1)^+ = \int_{r^1}^{\infty}(x^1 - r^1)f(x^1)dx^1$. And for the other batches $E(X^2 - r^2)^+ = \int_{r^2}^{\infty}(x^2 - r^2)f(x^2)dx^2$. Hence, in a cycle, the buyer's shortage cost is given by.

$$E(X^1 - r^1)^+ + (m-1)E(X^2 - r^2)^+ \quad (1)$$

For bi-level reorder point system, the expected net inventory level for the first batch just before an order arrival is equal to $E[(X^1 - r^1)^- I_{0<X^1<r^1}] - E(X^1 - r^1)^+$ and the expected net inventory level at the beginning of the cycle, given that γQ items are defective in an arriving order of size Q, equals $Q(1-\gamma) + E[(X^1 - r^1)^- I_{0<X^1<r^1}] - E(X^1 - r^1)^+$. For the other batches, expected net inventory level for the first batch just before an order arrival is equal to $E[(X^2 - r^2)^- I_{0<X^2<r^2}] - E(X^2 - r^2)^+$ and the expected net inventory level at the beginning of the cycle, given that γQ items are defective in an arriving order of size Q, equals $Q(1-\gamma) + E[(X^2 - r^2)^- I_{0<X^2<r^2}] - E(X^2 - r^2)^+$. Hence, the total holding cost per cycle is.

$$h_{b1}\left\{\frac{Q(1-\gamma)}{D}\left(\frac{Q(1-\gamma)}{2} + E[(X^1 - r^1)^- I_{0<X^1<r^1}] - E(X^1 - r^1)^+\right) + \frac{(m-1)Q(1-\gamma)}{D}\left(\frac{Q(1-\gamma)}{2} + E[(X^2 - r^2)^- I_{0<X^2<r^2}] - E(X^2 - r^2)^+\right)\right\} \quad (2)$$

The buyer's average inventory of defective items per unit time can be obtained as follows. The number of defective items in each successive m shipment is γQ and the screening period is Q/λ. Thus, the total inventory of defective item in each shipment is $\gamma Q^2/\lambda$. Hence, the buyer's average inventory of defective items per unit time is

$$\left[\frac{m\gamma Q^2}{\lambda}\right] / \left(\frac{m(1-\gamma)Q}{D}\right) = \frac{\gamma m Q D}{\lambda(1-\gamma)} \quad (3)$$

Hence, considering the buyer's cycle length, $m(1-\gamma)Q/D$, the buyer's cycle cost is given by:

$$(C|\gamma) = mut + A + m\left[(a_i + b_iQ)(t_{i-1} - t) + \sum_{j=1}^{i-1}(a_j + b_jQ)(NT_j - MT_j)\right] + h_{b1}\left\{\frac{Q(1-\gamma)}{D}\left(\frac{Q(1-\gamma)}{2} + E[(X^1 - r^1)^- I_{0<X^1<r^1}] - E(X^1 - r^1)^+\right) + \frac{(m-1)Q(1-\gamma)}{D}\left(\frac{Q(1-\gamma)}{2} + E[(X^2 - r^2)^- I_{0<X^2<r^2}] - E(X^2 - r^2)^+\right)\right\} + \frac{h_{b2}\gamma m Q^2}{\lambda} + \pi[E(X^1 - r^1)^+ + (m-1)E(X^2 - r^2)^+] + S^c mQ, \;\; t\epsilon[t_i, t_{i-1}] \quad (4)$$

Therefore, the buyer's expected inventory cost per cycle is as follows:

$$E(C|\gamma) = mut + A + m\left[(a_i + b_iQ)(t_{i-1} - t) + \sum_{j=1}^{i-1}(a_j + b_jQ)(NT_j - MT_j)\right] + \frac{h_{b1}mQ}{2D}\int_0^1(1-\gamma)^2 f(\gamma)d\gamma + h_{b1}\left\{\frac{Q(1-M_\gamma)}{D}(E[(X^1 - r^1)^- I_{0<X^1<r^1}] - E(X^1 - r^1)^+) + \frac{(m-1)Q(1-M_\gamma)}{D}(E[(X^2 - r^2)^- I_{0<X^2<r^2}] - E(X^2 - r^2)^+)\right\} + \frac{h_{b2}mM_\gamma Q^2}{\lambda} + \pi[E(X^1 - r^1)^+ + (m-1)E(X^2 - r^2)^+] + S^c mQ, \quad t\epsilon[t_i, t_{i-1}] \quad (5)$$

As mentioned earlier, the buyer's expected length of the cycle time is $E(T|\gamma) = mQ(1-M_\gamma)/D$. Hence, using the result of a basic theorem from renewal reward processes (Ross [21]), the expected annual cost can be computed as the expected cost per cycle divided by expected cycle time:

$$TEC_b = \left[\frac{\theta}{\delta}Ln\left(\frac{A_0}{A}\right) + \frac{DA}{mQ(1-M_\gamma)}\right] + \frac{D}{mQ(1-M_\gamma)}\left\{mut + A + m\left[(a_i + b_iQ)(t_{i-1} - t) + \sum_{j=1}^{i-1}(a_j + b_jQ)(NT_j - MT_j)\right]\right\} + \frac{h_{b1}Q}{2(1-M_\gamma)}\int_0^1(1-\gamma)^2 f(\gamma)d\gamma + \frac{h_{b1}}{m}\left(E\left[\left(X^1 - r^1\right)^- I_{0<X^1<r^1}\right] - E\left(X^1 - r^1\right)^+\right) + \frac{h_{b1}(m-1)}{m}\left(E\left[\left(X^2 - r^2\right)^- I_{0<X^2<r^2}\right] - E\left(X^2 - r^2\right)^+\right) + \frac{h_{b2}M_\gamma QD}{\lambda(1-M_\gamma)} + \frac{D\pi}{mQ(1-M_\gamma)}[E(X^1 - r^1)^+ + (m-1)E(X^2 - r^2)^+] + \frac{S^c D}{1-M_\gamma}, A\epsilon(0, A_0], t\epsilon[t_i, t_{i-1}] \quad (6)$$

As mentioned in assumption 14, the demand during the lead time for the first batch is a mixture of normally distributed with means $\mu_1 L(pQ, s, t)$, $\mu_2 L(pQ, s, t)$, and standard deviation $\sigma\sqrt{L(pQ, s, t)}$ and for the jth batch, $j = 2, \dots, m$, with means $\mu_1 L(t)$, $\mu_2 L(t)$ and $\sigma\sqrt{L(t)}$. Therefore, the safety stock (SS), can be expressed as follows

$$SS = \sigma\sqrt{s + t + pQ}\left\{\frac{r_1^1 + k_1(1-\alpha)}{\sqrt{1 + k_1^2\alpha(1-\alpha)}}\right\} \quad (7)$$

The safety stock also can be expressed as follows.

$$SS = \sigma\sqrt{t}\left\{\frac{r_1^2 + k_1(1-\alpha)}{\sqrt{1 + k_1^2\alpha(1-\alpha)}}\right\} \quad (8)$$

According to (Hsiao, 2008), From Eqs. (7) and (8), we have

$$\sigma\sqrt{s + t + pQ}\left\{\frac{r_1^1 + k_1(1-\alpha)}{\sqrt{1 + k_1^2\alpha(1-\alpha)}}\right\} = \sigma\sqrt{t}\left\{\frac{r_1^2 + k_1(1-\alpha)}{\sqrt{1 + k_1^2\alpha(1-\alpha)}}\right\} \quad (9)$$

The expected shortage of the first batch shipment is given as:

$$E(X^1 - r^1)^+ = \int_{r^1}^{\infty}(x^1 - r^1)f(x^1)dx^1 = \sigma\sqrt{t + s + pQ}\,\psi(r_1^1, r_2^1, \alpha) \quad (10)$$

For batches $2, \dots, m$, the expected shortage amount is:

$$E(X^2 - r^2)^+ = \int_{r^2}^{\infty} (x^2 - r^2) f(x^2) dx^2 = \sigma\sqrt{t}\,\psi(r_1^2, r_2^2, \alpha) \tag{11}$$

Hence, the buyer's total expected cost per unit time is reduced to

$$TEC_b = \left[\frac{\theta}{\delta} Ln\left(\frac{A_0}{A}\right) + \frac{DA}{mQ(1-M_\gamma)}\right] + \frac{D}{mQ(1-M_\gamma)}\left\{mut + A + m\left[a_i(t_{i-1} - t) + \sum_{j=1}^{i-1} a_j\left(NT_j - MT_j\right)\right]\right\} + \frac{h_{b1}}{m}\sigma\sqrt{s+t+pQ}\left\{\alpha\left[r_1^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma} + (1-\alpha)k_1\right) - \phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma} + (1-\alpha)k_1\right)\right] + (1-\alpha)\left[r_2^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma} - \alpha k_1\right) - \phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma} - \alpha k_1\right)\right]\right\} + \frac{(m-1)h_{b1}}{m}\sigma\sqrt{t}\left\{\alpha\left[r_1^1\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma} + (1-\alpha)k_1\right) - \phi\left(\frac{\mu_*\sqrt{t}}{\sigma} + (1-\alpha)k_1\right)\right] + (1-\alpha)\left[r_2^1\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma} - \alpha k_1\right) - \phi\left(\frac{\mu_*\sqrt{t}}{\sigma} - \alpha k_1\right)\right]\right\} + \frac{h_{b1}Q}{2(1-M_\gamma)}\int_0^1 (1-\gamma)^2 f(\gamma) d\gamma + \frac{h_{b2}M_\gamma QD}{\lambda(1-M_\gamma)} + \frac{D\pi}{mQ(1-M_\gamma)}\left[\sigma\sqrt{t+s+pQ}\,\psi(r_1^1, r_2^1, \alpha) + (m-1)\sigma\sqrt{t}\,\psi(r_1^2, r_2^2, \alpha)\right] + \frac{S^c D}{1-M_\gamma} + \frac{D}{1-M_\gamma}\left[b_i(t_{i-1} - t) + \sum_{j=1}^{i-1} b_j\left(NT_j - MT_j\right)\right], A\epsilon(0, A_0], t\epsilon[t_i, t_{i-1}] \tag{12}$$

With today's high cost of land acquisition in most societies, most of the inventory systems have limited storage space to stock goods. Therefore, for the proposed inventory system, it is assumed that maximum permissible storage space is limited. The proposed constraint is probabilistic since the buyer's maximum inventory level is a random variable. The mentioned probabilistic constraint can be expressed by

$$P\{f[Q(1-\gamma) + E[(X-r)^- I_{0<X<r}] - E(X-r)^+] \le F\} \ge \varphi \tag{13}$$

The above constraint forces the probability that the total used space is within maximum permissible storage space to be no smaller than φ. It is problematic to solve the constrained inventory system when the space constraint is written as (13). Hence, by using the chance-constrained programming technique, which is proposed by (Charnes & Cooper, 1959) and considering Markov inequality, the random constraint for a mixture of normal distributions is converted to the crisp one which is given by:

$$\varphi_1 Q + \sigma\sqrt{s+t+pQ}\left\{\alpha\left[r_1^1\Phi\left(\frac{\mu_*\sqrt{s+t+pQ}}{\sigma} + (1-\alpha)k_1\right) - \phi\left(\frac{\mu_*\sqrt{s+t+pQ}}{\sigma} + (1-\alpha)k_1\right)\right] + (1-\alpha)\left[r_2^1\Phi\left(\frac{\mu_*\sqrt{s+t+pQ}}{\sigma} - \alpha k_1\right) - \phi\left(\frac{\mu_*\sqrt{s+t+pQ}}{\sigma} - \alpha k_1\right)\right]\right\} - QM_\gamma - \frac{F}{f} \le 0 \tag{14}$$

And

$$\varphi_2 Q + \sigma\sqrt{t}\left\{\alpha\left[r_1^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma} + (1-\alpha)k_1\right) - \phi\left(\frac{\mu_*\sqrt{t}}{\sigma} + (1-\alpha)k_1\right)\right] + (1-\alpha)\left[r_2^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma} - \alpha k_1\right) - \phi\left(\frac{\mu_*\sqrt{t}}{\sigma} - \alpha k_1\right)\right]\right\} - QM_\gamma - \frac{F}{f} \le 0 \tag{15}$$

3.2 Vendor's total expected cost per unit time

The initial stock in the system, QD/P, is the amount of inventory required by the buyer during the protection period of the first shipment q. As soon as the production run is started, the total stock increases at a rate of $(P - D)$ until the complete batch quantity, mQ, has been manufactured. Hence, the total inventory level per unit time for the vendor can be calculated as follows.

$$(C|\gamma)_v = mQ\left[\frac{Q}{P} + (m-1)\frac{(1-\gamma)Q}{D} - \frac{m^2Q^2}{2P}\right] - \left[\frac{Q^2(1-\gamma)}{D}\left(1 + 2 + \dots + (m-1)\right)\right] = \frac{mQ^2}{P} + \frac{m(m-1)Q^2(1-\gamma)}{2D} - \frac{m^2Q^2}{2P} \tag{16}$$

Hence, the vendor's cycle cost is given by:

$$E(C|\gamma)_v = as + \theta\left[c_{si}(s_{i-1} - s) + \sum_{j=1}^{i} c_{sj}\left(NS_j - MS_j\right)\right] + h_V\left[\frac{mQ^2}{P} + \frac{m(m-1)(1-M_\gamma)Q^2}{2D} - \frac{m^2Q^2}{2P}\right] + W^c M_\gamma mq, \quad s\epsilon[s_i, s_{i-1}] \tag{17}$$

Considering the vendor's expected length of the cycle time is $E(T|\gamma) = mQ(1 - M_\gamma)/D$ and renewal reward processes, the vendor's expected total cost per unit time is computed as given below

$$TEC_v = \frac{D}{mQ(1-M_\gamma)}\left\{as + \left[c_{si}(s_{i-1} - s) + \sum_{j=1}^{i-1} c_{sj}\left(NS_j - MS_j\right)\right]\right\} + \frac{h_V DQ}{1-M_\gamma}\left[\frac{1}{P} + \frac{(m-1)(1-M_\gamma)}{2D} - \frac{m}{2P}\right] + \frac{W^c M_\gamma D}{1-M_\gamma}, \quad s\epsilon[s_i, s_{i-1}] \tag{18}$$

3.3 Joint total expected cost per unit time

Once the buyer and vendor have built up a long-term strategic partnership, they can jointly determine the best policy for both parties. Accordingly, the joint total expected cost per unit time can be obtained as the sum of the buyer's and the vendor's total expected costs per unit time. That is,

$$JEAC(Q,A,r^1,r^2,s,t,m) = \left[\frac{\theta}{\delta}Ln\left(\frac{A_0}{A}\right)+\frac{DA}{mQ(1-M_\gamma)}\right]+\frac{h_{b1}}{m}\sigma\sqrt{s+t+pQ}\left\{\alpha\left[r_1^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)-\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)\right]+(1-\alpha)\left[r_2^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)\right]\right\}+\frac{(m-1)h_{b1}}{m}\sigma\sqrt{t}\left\{\alpha\left[r_1^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+(1-\alpha)k_1\right)-\phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+(1-\alpha)k_1\right)\right]+(1-\alpha)\left[r_2^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)\right]\right\}+\frac{h_{b1}Q}{2(1-M_\gamma)}\int_0^1(1-\gamma)^2f(\gamma)d\gamma+\frac{h_{b2}M_\gamma QD}{\lambda(1-M_\gamma)}+\frac{D\pi}{mQ(1-M_\gamma)}\left[\sigma\sqrt{t+s+pQ}\psi(r_1^1,r_2^1,\alpha)+(m-1)\sigma\sqrt{t}\psi(r_1^2,r_2^2,\alpha)\right]+\frac{S^cD}{1-M_\gamma}+\frac{W^cM_\gamma D}{1-M_\gamma}+\frac{D}{mQ(1-M_\gamma)}\left\{mut+A+as+\left[c_{si}(s_{i-1}-s)+\sum_{j=1}^{i-1}c_{sj}(NS_j-MS_j)\right]+m\left[a_i(t_{i-1}-t)+\sum_{j=1}^{i-1}a_j(NT_j-MT_j)\right]\right\}+\frac{D}{1-M_\gamma}\left[b_i(t_{i-1}-t)+\sum_{j=1}^{i-1}b_j(NT_j-MT_j)\right]+\frac{h_vDQ}{1-M_\gamma}\left[\frac{1}{P}+\frac{(m-1)(1-M_\gamma)}{2D}-\frac{m}{2P}\right]$$

Subject to:

$$\sigma\sqrt{s+t+pQ}[r_1^1+k_1(1-\alpha)]-\sigma\sqrt{t}[r_1^2+k_1(1-\alpha)]=0$$

$$\gamma Q+\sigma\sqrt{s+t+pQ}\left\{\alpha\left[r_1^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)-\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)\right]+(1-\alpha)\left[r_2^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)\right]\right\}-QM_\gamma-\frac{F}{f}\le 0$$

$$\gamma Q+\sigma\sqrt{t}\left\{\alpha\left[r_1^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+(1-\alpha)k_1\right)-\phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+(1-\alpha)k_1\right)\right]+(1-\alpha)\left[r_2^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)\right]\right\}-QM_\gamma-\frac{F}{f}\le 0$$

$$\text{Over } Q,r^1,r^2\ge 0, A\epsilon(0,A_0], t\epsilon[t_i,t_{i-1}], s\epsilon[s_i,s_{i-1}], m>0 \text{ integer} \quad (19)$$

The above model (19) can be solved with the Lagrange multiplier method as given below:

$$JEAC(Q,A,r^1,r^2,s,t,m,\lambda_1,\lambda_2,\lambda_3) = \left[\frac{\theta}{\delta}Ln\left(\frac{A_0}{A}\right)+\frac{DA}{mQ(1-M_\gamma)}\right]+\frac{h_{b1}}{m}\sigma\sqrt{s+t+pQ}\left\{\alpha\left[r_1^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)-\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)\right]+(1-\alpha)\left[r_2^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)\right]\right\}+\frac{(m-1)h_{b1}}{m}\sigma\sqrt{t}\left\{\alpha\left[r_1^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+(1-\alpha)k_1\right)-\phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+(1-\alpha)k_1\right)\right]+(1-\alpha)\left[r_2^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)\right]\right\}+\frac{h_{b1}Q}{2(1-M_\gamma)}\int_0^1(1-\gamma)^2f(\gamma)d\gamma+\frac{h_{b2}M_\gamma QD}{\lambda(1-M_\gamma)}+\frac{D\pi}{mQ(1-M_\gamma)}\left[\sigma\sqrt{t+s+pQ}\psi(r_1^1,r_2^1,\alpha)+(m-1)\sigma\sqrt{t}\psi(r_1^2,r_2^2,\alpha)\right]+\frac{S^cD}{1-M_\gamma}+\frac{W^cM_\gamma D}{1-M_\gamma}+\frac{D}{mQ(1-M_\gamma)}\left\{mut+as+\left[c_{si}(s_{i-1}-s)+\sum_{j=1}^{i-1}c_{sj}(NS_j-MS_j)\right]+m\left[a_i(t_{i-1}-t)+\sum_{j=1}^{i-1}a_j(NT_j-MT_j)\right]\right\}+\frac{D}{(1-M_\gamma)}\left[b_i(t_{i-1}-t)+\sum_{j=1}^{i-1}b_j(NT_j-MT_j)\right]+\frac{h_vDQ}{1-M_\gamma}\left[\frac{1}{P}+\frac{(m-1)(1-M_\gamma)}{2D}-\frac{m}{2P}\right]+\lambda_1\sigma\sqrt{s+t+pQ}[r_1^1+k_1(1-\alpha)]-\lambda_1\sigma\sqrt{t}[r_1^2+k_1(1-\alpha)]+\lambda_2\sigma\sqrt{s+t+pQ}\left\{\alpha\left[r_1^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)-\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)\right]+(1-\alpha)\left[r_2^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)\right]\right\}+\lambda_2Q(\varphi_1-M_\gamma)-\lambda_2\frac{F}{f}+\lambda_3\sigma\sqrt{t}\left\{\alpha\left[r_1^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+(1-\alpha)k_1\right)-\phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+(1-\alpha)k_1\right)\right]+(1-\alpha)\left[r_2^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)\right]\right\}+\lambda_3Q(\varphi_2-M_\gamma)-\lambda_3\frac{F}{f} \quad (20)$$

Where λ_1 is free in sign and λ_2 and λ_3 are nonnegative variables. To solve the above nonlinear programming problem, this study temporarily ignores the constraint $0\le A\le A_0$ and relaxes the integer requirement on m(the number of shipments from the vendor to the buyer during a cycle). It can be shown that for fixed $Q,A,r^1,r^2,s,t,m,\lambda_1,\lambda_2$, the optimal setup and transportation time occur at the end of points of interval $s\epsilon[s_i,s_{i-1}]$ and $t\epsilon[t_i,t_{i-1}]$ respectively (Chang et al., 2006). This result simplifies the search for the optimal solution to this inventory problem considerably. Therefore, the Kuhn-Tucker necessary conditions for minimization of the function (20) are as follows:

$$\frac{\partial JEAC}{\partial Q}=0, \frac{\partial JEAC}{\partial r^1}=0, \frac{\partial JEAC}{\partial r^2}=0, \frac{\partial JEAC}{\partial A}=0 \quad (21)$$

$$\sigma\sqrt{s+t+pQ}[r_1^1+k_1(1-\alpha)]-\sigma\sqrt{t}[r_1^2+k_1(1-\alpha)]=0 \quad (22)$$

$$\lambda_2\left\{\varphi_1Q+\sigma\sqrt{s+t+pQ}\left\{\alpha\left[r_1^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)-\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)\right]+(1-\alpha)\left[r_2^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)\right]\right\}-QM_\gamma-\frac{F}{f}\right\}=0 \quad (23)$$

$$\lambda_3\left\{\gamma Q+\sigma\sqrt{t}\left\{\alpha\left[r_1^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+(1-\alpha)k_1\right)-\phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+(1-\alpha)k_1\right)\right]+(1-\alpha)\left[r_2^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)\right]\right\}-QM_\gamma-\frac{F}{f}\right\}=0 \quad (24)$$

Solving Eqs. in (21) respectively produces:

$$Q=\sqrt{\frac{\frac{D}{m(1-M_\gamma)}\{mut+A+as+mU(t)+CS(s)+\sigma\sqrt{t+s+pQ}\psi(r_1^1,r_2^1,\alpha)+(m-1)\sigma\sqrt{t}\psi(r_1^2,r_2^2,\alpha)\}}{H_v+H_{b_1,b_2,\lambda_1,\lambda_2}+\frac{Dp\pi\sigma\psi(r_1^1,r_2^1,p)}{2mQ(1-M_\gamma)\sqrt{s+t+pQ}}}} \quad (25)$$

Where

$$U(t)=a_i(t_{i-1}-t)+\sum_{j=1}^{i-1}a_j(NT_j-MT_j) \quad (26)$$

$$CS(s)=c_{si}(s_{i-1}-s)+\sum_{j=1}^{i-1}c_{sj}(NS_j-MS_j) \quad (27)$$

$$H_v=\frac{h_vD}{1-M_\gamma}\left[\frac{1}{P}+\frac{(m-1)(1-M_\gamma)}{2D}-\frac{m}{2P}\right] \quad (28)$$

$$H_{b_1,b_2,\lambda_1,\lambda_2}=\frac{h_{b1}}{2(1-M_\gamma)}\int_0^1(1-\gamma)^2f(\gamma)d\gamma+\frac{h_{b2}M_\gamma D}{\lambda(1-M_\gamma)}+\frac{\left(\frac{h_{b1}}{m}+\lambda_1+\lambda_2\right)p\sigma}{2\sqrt{s+t+pQ}}\left\{\alpha\left[r_1^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)-\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)\right]+(1-\alpha)\left[r_2^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)\right]\right\}+\frac{\left(\frac{h_{b1}}{m}+\lambda_1+\lambda_2\right)\mu_*p}{2}\left[\alpha\left(r_1^1+\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)+(1-\alpha)\left(r_2^1+\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)\right]+\frac{\lambda_1\sigma p[r_1^1+k_1(1-\alpha)]}{2\sqrt{s+t+pQ}}+\lambda_2(\varphi_1-M_\gamma)+\lambda_3(\varphi_2-M_\gamma) \quad (29)$$

And

$$\left(\frac{h_{b1}}{m}+\lambda_2\right)\left[\alpha\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)+(1-\alpha)\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)\right]+\lambda_1\sigma\sqrt{t+s+pQ}=\frac{D\pi}{mQ(1-M_\gamma)}\left(1-F_*(r^1)\right) \quad (30)$$

$$\left(\frac{(m-1)h_{b1}}{m}+\lambda_3\right)\sigma\sqrt{t}\left[\alpha\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+(1-\alpha)k_1\right)+(1-\alpha)\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)\right]-\lambda_1\sigma\sqrt{t}=\frac{D\pi(m-1)}{mQ(1-M_\gamma)}\left(1-F_*(r^2)\right) \quad (31)$$

$$A=\frac{\theta mQ(1-M_\gamma)}{D\delta} \quad (32)$$

Where $F_*(r^1)=\alpha\Phi(r_1^1)+(1-\alpha)\Phi(r_2^1)$ and $F_*(r^2)=\alpha\Phi(r_1^2)+(1-\alpha)\Phi(r_2^2)$. On the other hand, for fixed s,t and m, it can be shown that $JEAC(Q,A,r^1,r^2,s,t,m,\lambda_1,\lambda_2,\lambda_3)$ is convex in (A,r^1,r^2) since the objective function, $JEAC(Q,A,r^1,r^2,s,t,m)$, is convex in (A,r^1,r^2) by examining second-order sufficient condition and also the constraints are linear in (A,r^1,r^2); however, may not be convex in (Q,A,r^1,r^2). Therefore, the following algorithm can be used to find an approximate solution to the above problem.

3.4 Solution Procedure

Step1. Set $m=1$.

Step2. Compute the intersection points Q^s of the crash cost lines $c_i=a_i+b_iQ$ and $c_j=a_j+b_jQ$, for all i and j, where $a_i>a_j$, $b_i<b_j$, $i\neq j$ and $i,j=1,2,\dots,n^t$. Arrange these intersection points such that ${Q_1}^s<{Q_2}^s<\cdots<{Q_w}^s$ and let ${Q_0}^s=0$, ${Q_{w+1}}^s=\infty$.

Step3. Rearrange c_i such that $c_1\le c_2\le\cdots\le c_{n^t}, j=1,2,\dots,w$, for the order quantity range $({Q_{j-1}}^s,{Q_j}^s)$.

Step4. For each t_i and s_z, $i=0,1,\dots,n^t, z=0,1,\dots,n^s$, perform Step 4-1 to Step 4-10.

Step4-1. Set $\lambda_2=0$ and $\lambda_3=0$ and solve the problem without space constraint.

Step4-2. Compute $Q_{iz}^1=\sqrt{\frac{D}{m(1-M_\gamma)}\{mut+A+as+mU(t)+CS(s)\}/H_v}$.

Step4-3. Find A_{iz}^1 from Eq. (32).

Step4-4. Find $r_{iz}^{1^1}$, $r_{iz}^{2^1}$ in terms of λ_1 from Eqs. (31) and (30).

Setp4-5. Setting the values $Q_{iz}^1, r_{iz}^{1^1}$ and $r_{iz}^{2^1}$ in Eq. (22) and find $\lambda_{1_{iz}}^1$.

Step4-6. Compute Q_{iz}^2 from (25) using $A_{iz}^1, r_{iz}^{1^1}, r_{iz}^{2^1}$ and $\lambda_{1_{iz}}^1$.

Step4-7. Repeat Step 4-2 to Step 4-6 until no changes occur in the values of Q_{iz}, A_{iz}, r_{iz}^1 and r_{iz}^2.

Step4-8. Check whether $A_{iz}<A_0$ and $Q_i\in[{Q_{j-1}}^s,{Q_j}^s]$:

Step4-8-1. If $A_{iz} < A_0$ and $Q_{iz} \in [{Q_{j-1}}^s, {Q_j}^s]$, then the solution found in Step 4-2 to Step 4-7 is optimal for given t_i and s_z go to step (4).
Step4-8-2. If $A_{iz} \geq A_0$, for given t_i and s_z, set $A_{iz} = A_0$ and obtain $Q_{iz}, r_{iz}^1, r_{iz}^2, \lambda_{1_{iz}}$ by solving Eqs. (25), (30), (31) and (22) iteratively until convergence.
Step4-8-3. If $Q_{iz} \leq {Q_{j-1}}^s$, let $Q_{iz} = {Q_{j-1}}^s$ and if ${Q_j}^s \leq Q_{iz}$ let ${Q_j}^s = Q_{iz}$. Using Q_{iz} as a constant, obtain $A_{iz}, r_{iz}^1, r_{iz}^2$ and $\lambda_{1_{iz}}$ by solving Eqs. (30) to (32) and (22) iteratively until convergence.
Step4-9. If the solution for $Q_{iz}, A_{iz}, r_{iz}^1, r_{iz}^2$ and $\lambda_{1_{iz}}$ satisfies the space constraint from model (19), then go to step (5) otherwise go to step (4-10).
Step4-10.If the solution for $Q_{iz}, A_{iz}, r_{iz}^1, r_{iz}^2$ and $\lambda_{1_{iz}}$ don't satisfy the space constraint, determine the new $Q_{iz}, A_{iz}, r_{iz}^1, r_{iz}^2$, $\lambda_{1_{iz}}$, $\lambda_{2_{iz}}$ and $\lambda_{3_{iz}}$ by a procedure similar to given In Step 4 then go to Step 5.
Step5. Find min $JTEC(Q_{iz}, A_{iz}, r_{iz}^1, r_{iz}^2, t_i, s_z) = JTEC(Q^m, A^m, {r^1}^m, {r^2}^m, t^m, s^m)$ for $i = 0,1,\dots,n^t, z = 0,1,\dots,n^s$.
Step6. Set $m = m + 1$, and repeat Steps 2 to5 to get $JTEC(Q^m, A^m, {r^1}^m, {r^2}^m, t^m, s^m)$.
Step7. If $JTEC(Q^m, A^m, {r^1}^m, {r^2}^m, t^m, s^m, m) \leq JTEC(Q^{m-1}, A^{m-1}, {r^1}^{m-1}, {r^2}^{m-1}, t^{m-1}, s^{m-1}, m-1)$, then go to step 6, otherwise go to step 8.
Step8. Set $(Q^*, A^*, {r^1}^*, {r^2}^*, t^*, s^*, m^*) = (Q^m, A^m, {r^1}^m, {r^2}^m, t^m, s^m, m)$, then $(Q^*, A^*, k_1^*, t^*, s^*, m^*)$ is the optimal solution and $JTEC\ (Q^*, A^*, {r^1}^*, {r^2}^*, t^*, s^*, m^*)$ is the minimum joint expected annual cost.

4. Numerical example

To illustrate the behavior of the model developed in this paper, let us consider an inventory problem with the following data: $D = 624$ units per year, h_{b_1} =10\$ per unit per year, h_{b_2} =5\$ per unit per year, h_v =3\$ per unit per year, A_0 =50\$ per order, $\lambda = 5000$ per year, S^c =1\$ per unit, W^c =10\$ per unit, a =1000\$ per week, u = 7\$ per week, p =1/62.5 week per unit, σ= 15 units per week, π =70\$ per unit per year, f= 3 M^2 per unit, F =400 M^2 ,$\varphi_1 = 0.99$, $\varphi_2 = 0.99$, $\theta = 0.1$ and $\delta = 1/700$.
Defective rate γ in an order lot has a Beta distribution function with parameters $a = 20$ and $b = 80$; that is, the p.d.f. of γ is given by:

$$g(\gamma) = \frac{\Gamma(60)}{\Gamma(20)\Gamma(40)}\gamma^9(1-\gamma)^{39},\quad 0 < \gamma < 1$$

Therefore, we have:

$$M_\gamma = \frac{a}{a+b} = 0.2 \quad \text{and} \quad E(\gamma^2) = \frac{a(a+1)}{(a+b)(a+b+1)} = 0.043$$

Moreover, we consider 1 year= 48 weeks. The lead time has three components with data shown in Table 1.

Table 1. Lead time data

Lead time component i	1	2	3
Normal duration T_i (days)	20	20	16
Minimum duration t_i (days)	6	6	9
Unit fixed crash cost a_i (\$/day)	0.5	1.3	5.1
Unit variable crash cost b_i (\$/unit/day)	0.012	0.004	0.0012

Table 2's data are first used to evaluate the intersection points, order quantity rage interval, and component crash priorities in each interval. Table 2 shows the crash sequence corresponding to each order quantity range.

Table 2. The values of Q^s, order quantity ranges and crash sequence

Inspection points (Q^s)	Order quantity range	Crash sequence of components
100	(0, 100]	1, 2, 3
426	(100, 426]	2, 1, 3
1357	(426, 1357]	2, 3, 1
---	(1357, ∞)	3, 2, 1

Setup times and their respective crashing costs are tabulated in Table 3.

Table 3. Setup time data

Setup time component i	1	2	3
Normal duration NS_i (days)	0.14	0.14	0.07
Minimum duration MT_i (days)	0.105	0.105	0.049
Unit fixed crash cost c_{si} (\$/day)	2000	3000	5000

We first assume the model without space constraint and solve the case when $\alpha = 0.0, 0.3, 0.8, 1.0$ and $k_1 = 0.7$. Applying the proposed algorithm yields the optimal solutions as tabulated in Table 4.

Table 4. Summary of the results for the model without space constraint

α	Q	A	r^1	r^2	m	s	t	$JTEC$
0.0	99	26.70	141	120	3	0.04	4	4597.26
0.3	99	26.74	145	124	3	0.05	4	4632.24
0.8	99	26.67	143	122	3	0.05	4	4610.00
1.0	99	26.70	141	120	3	0.05	4	4597.26

Results of optimal decisions show that for fixed value of m, t and s, with an augment in α, the two optimal reorder point for different batches increase. We also observe that when $\alpha = 0$ or 1, the model considers only one kind of customers' demand; when $0 \leq \alpha \leq 1$, the model considers two kinds of customers' demand. It implies that the minimum joint expected annual cost with two kinds of customers' demand is larger than the minimum expected annual cost with one kind of customers' demand. Thus, the minimum joint expected annual cost increases as the distance between α and 0 (or 1) increased.
Then, we assume space-constrained model and solve the case when $\alpha = 0.0, 0.3, 0.8, 1.0$ and $k_1 = 0.7$. Utilizing the presented algorithm, optimal decisions are obtained which are tabulated in Table 5.

Table 5. Summary of the results for the model with space constraint

α	Q	A	r^1	r^2	m	s	t	$JTEC$
0.0	89	24.01	137	118	3	0.04	4	4606.91
0.3	86	23.19	140	121	3	0.05	4	4649.88
0.8	87	23.64	139	120	3	0.05	4	4622.40
1.0	89	24.01	137	118	3	0.05	4	4606.91

Similar to the unconstrained model, for a fixed value of m, t, and s, with an augment in α, the two optimal reorder points for different batches increase. Also, the optimum joint expected annual cost for two kinds of customers' demand is larger than one kind of customer demand.

In the following, we conduct a one-way sensitivity analysis to assess the impact of the problem parameters on the joint expected total cost per unit time. This numerical experiment is carried out by varying one parameter at a time and keeping the remaining ones at their base values. Table 6 shows the values for different problem parameters to be used in the sensitivity analysis.

Table 6. Experimental values for the example parameters

Parameter	Base value	Experimental values

D	624	500	550	624	650	700	750
$\gamma(a,b)$	$\gamma(10,40)$	$\gamma(10,70)$	$\gamma(10,50)$	$\gamma(10,40)$	$\gamma(10,30)$	$\gamma(10,20)$	$\gamma(25,60)$
σ	15	5	10	15	20	25	30
h_v	3	1	2	3	4	5	6
F	400	300	350	400	450	500	550
π	70	50	60	70	80	90	100
h_{b1}	10	5	7	10	12	14	15
h_{b2}	5	2	4	5	7	9	10
A_0	50	20	30	50	60	80	100
S^c	1	0.5	0.75	1	1.25	1.5	1.75
δ	1/700	1/400	1/600	1/700	1/1000	1/1200	1/1400
a	1000	500	750	1000	1250	1500	1750
u	7	4	6	7	9	11	13

Figure 1. displays graphically the results of the sensitivity study as a tornado diagram, which shows how the joint expected total cost per unit time changes while the problem parameters are independently varied from their low to high values. The length of each bar in the diagram represents the extent to which the expected joint total cost per unit time is sensitive to the bar's corresponding problem parameter. It can be observed from Fig. 1 that the problem parameters with the greatest impact on the model's expected cost is defective rate. With other parameters held at their base values, when defective rate is varied from $\gamma(10,70)$ to $\gamma(2,6)$, the value of the joint expected per unit time changed from 3517 to 5849. This shows a larger amount of defective rate can be highly affect the joint expected total cost. Other problem parameters which have largest impact on joint expected cost are average demand per unit time and buyer's demand standard deviation. Therefore, the inventory decision maker must carefully estimate the values of these parameters since they have most significant effect on the model's cost.

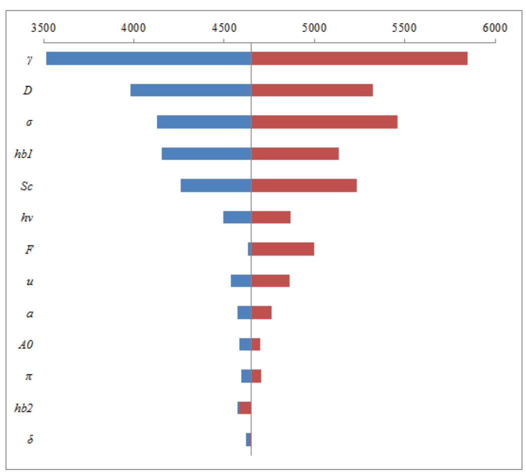

Figure 1. Sensitivity analysis results

5. Conclusion

The purpose of this paper is to propose a multi-reorder level inventory-production model in which the buyer's LTD follows the mixture of distributions. The paper assumes the buyer's maximum permissible storage space is limited and therefore adds a space constraint to the respective inventory system. It is also assumed that each lot received contains percentage defectives with a known probability density function. Lead time components and ordering cost are considered to be controllable. A Lagrangian method is utilized to solve the model, and a solution procedure is proposed to find optimal values. The behavior of the model is illustrated in numerical examples. Results of optimal decisions show that for a fixed value of m, t and s, with an augment in α, the two optimal reorder points for different batches increase. We also observe that when $\alpha = 0$ or 1, the model considers only one kind of customers' demand; when $0 \leq \alpha \leq 1$, the model considers two kinds of customers' demand. It implies that the minimum joint expected annual cost with two kinds of customers' demand is larger than the minimum expected annual cost with one kind of customer demand. Thus, the minimum joint expected annual cost increases as the distance between α and 0 (or 1) increased. To increase the scope of our analysis, the model presented in this article could be extended in several ways. For example, shortage cost can be calculated as a mixture of backorder and lost sales. Thus, with an increasing or a decreasing in a backorder rate, the optimal order quantity and reorder level may be higher or lower. Also, investigating on some other LTD approach such as gamma and lognormal distribution could be considered. Other kind of constraints such as budget constraint could be added to make the system closer to real environment.

References

Banerjee, A. (1986). Economic-lot-size model for purchaser and vendor. *Decis. Sci*, *17*, 292–311.

Chang, H.-C., Ouyang, L.-Y., Wu, K.-S., & Ho, C.-H. (2006). Integrated vendor–buyer cooperative inventory models with controllable lead time and ordering cost reduction. *European Journal of Operational Research*, *170*(2), 481–495.

Charnes, A., & Cooper, W. W. (1959). Chance-Constrained Programming. *Management Science*, *6*(1), 73–79.

Fazeli, S. S., Venkatachalam, S., Chinnam, R. B., & Murat, A. (2020). Two-Stage Stochastic Choice Modeling Approach for Electric Vehicle Charging Station Network Design in Urban Communities. *IEEE Transactions on Intelligent Transportation Systems*, 1–16.

Goyal, S. K. (1977). An integrated inventory model for a single supplier-single customer problem. *International Journal of Production Research*, *15*(1), 107–111.

Haksever, C., & Moussourakis, J. (2005). A model for optimizing multi-product inventory systems with multiple constraints. *International Journal of Production Economics*, *97*(1), 18–30.

Hariga, M. A. (2010). A single-item continuous review inventory problem with space restriction. *International Journal of Production Economics*, *128*(1), 153–158.

Ho, C.-H. (2009). A minimax distribution free procedure for an integrated inventory model with defective goods and stochastic lead time demand. *International Journal of Information and Management Sciences*, *20*, 161–171.

Hsiao, Y. C. (2008). A note on integrated single vendor single buyer model with stochastic demand and variable lead time. *International Journal of Production Economics*, *114*(1), 294–297.

Huang, C.-K. (2002). An integrated vendor-buyer cooperative inventory model for items with imperfect quality. *Production Planning & Control*, *13*(4), 355–361.

Lee, W. C., Wu, J. W., & Hou, W. Bin. (2004). A note on inventory model involving variable lead time with defective units for mixtures of distribution. *International Journal of Production Economics*, *89*(1), 31–44.

Liao, C., & Shyu, C. (1991). An Analytical Determination of Lead Time with Normal Demand. *International Journal of Operations & Production Management*, *11*(9), 72–78.

Lou, K.-R., & Wang, W.-C. (2013). A comprehensive extension of an integrated inventory model with ordering cost reduction and permissible delay in payments. *Applied Mathematical Modelling*, *37*(7), 4709–4716.

Moon, I., & Ha, B.-H. (2012). Inventory systems with variable capacity. *European Journal of Industrial Engineering*, *6*(1), 68–86.

Ouyang, L. Y., Wu, K. S., & Ho, C. H. (2004). Integrated vendor-buyer cooperative models with stochastic demand in controllable lead time. *International Journal of Production Economics*, *92*(3), 255–266.

Porteus, E. L. (1985). Investing in Reduced Setups in the EOQ Model. *Management Science*, *31*(8), 998–1010.

Ross, S. (1996). *Stochastic Processes* (2nd ed). Wiley.

Tahami, H., Mirzazadeh, A., Arshadi-Khamseh, A., & Gholami-Qadikolaei, A. (2016). A periodic review integrated inventory model for buyer's unidentified protection interval demand distribution. *Cogent Engineering*, *3*(1).

Tahami, H., Mirzazadeh, A., & Gholami-Qadikolaei, A. (2019). Simultaneous control on lead time elements and ordering cost for an inflationary inventory-production model with mixture of normal distributions LTD under finite capacity. *RAIRO-Oper. Res.*, *53*(4), 1357–1384.

Veinott, A. (1965). Optimal Policy for a Multi-Product, Dynamic, Nonstationary Inventory Problem. *Management Science*, *12*(3), 206–222.

Yahoodik, S., Tahami, H., Unverricht, J., Yamani, Y., Handley, H., & Thompson, D. (2020). Blink Rate as a Measure of Driver Workload during Simulated Driving. *Proceedings of the Human Factors and Ergonomics Society 2020 Annual Meeting, Chicago, IL.*

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