Three-phase Growth Model in Fibonacci Rabbits

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Abstract

Fibonacci numbers or Fibonacci sequences are numbers or sequences that are very popular in mathematics. In the perspective of Mathematical Demographics, Fibonacci numbers represent the number of populations of rabbit pairs at any time. We consider this population as a hypothetical population with unlimited growth. We observe the population structure based on the phases (stages) of rabbit growth. We begin the discussion by reviewing the Fibonacci number rows in relation to the perspective of the dynamics of population systems. We analysed the hypothetical population system as a population system consisting of three age groups. Next, we build mathematical equations to produce Fibonacci numbers. The equation is built as a matrix equation, involving the growth matrix. We compiled and proved several theorems. The results of the research related to the growth matrix and its relation to the demographic bonus we describe at the end of the discussion.

Keywords:
Fibonacci numbers, golden ratios, growth matrix, hypothetical populations, matrix equations.

1. Introduction

Fibonacci sequence numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, … The numbers on the Fibonacci sequence represent the number of pairs of rabbits at each time Fernando and Prabowo, 2019). Suppose for $n = 1$ we have 1 pair of male and female rabbits. For $n = 2$ the number of pairs of rabbits still remains 1. For the next time, the population of rabbits is always increasing. Thus, the Fibonacci number sequence can be used to study population growth in the fields of ecology, biology, demography and others (Supriatna, Carnia and Ndii, 2019).

Many examples of events in the universe that follow the Fibonacci sequence and its golden ratio, such as biological arrangements such as branching on trees, pineapple sprouts, ferns, phyllotaxis, flowering artichokes, that are not rounded and bract arrangement of pine cones. Sinha (2017) provides some additional example of using Fibonacci sequences in coding theory. Fibonacci sequences in various forms are widely applied in making security codes. Pirillo (2019) discusses Fibonacci Word and Sturmian Words, geometries construction that built from Fibonacci sequences, and presents some ideas in mathematics education. Schuster, Fichtner, and Sasso (2017) show that the amount of fatty acids has the potential to increase with their long chains, and the increase in numbers grows in the Fibonacci sequence obtained by cis/trans-isomerism. Schneider (2016) discuss continued fractions and nested radicals relations involving the Fibonacci sequence. Sacco (2019**)** models the relationship between the mechanical cycles of celestial bodies and Fibonacci numbers. The results obtained are useful for understanding the optimal way the solar system reaches its stability.

In demographic studies, we can probe deeper into the Fibonacci sequences that describe the growth of rabbit populations. The growth of rabbits with two phases results in the composition of a population with two age groups, namely the non-productive and productive age group. The growth of rabbits in three phases produces a population composition with three age groups, namely the age group of infants, adolescents and adults.

In the Fibonacci sequence, the pair of rabbits develops from infancy to adulthood and has offspring (Koshy). Thus, we can group Fibonacci rabbits for each time point based on the number of pairs of babies, juvenils and adults. As a result, we have three age groups for Fibonacci rabbits, namely the baby age group, the juvenil age group and the adult age group. Furthermore, we call this three-phase growth model.

We can also do other groupings. We make the growth of rabbits with three phases as two phases. The first phase is the unproductive phase. In this phase we enter a pair of rabbits who are still babies. The second phase is the productive phase. In this group we enter a pair of juvenil and adult rabbits. Thus we have two age groups namely the non-productive and productive age groups.

We will begin the discussion by reviewing the Fibonacci sequences from the perspective of population dynamics. Based on the division into three age groups, we will determine the number of pairs of rabbits, including infants, juvenils and adults at each time point. Based on the division into two age groups, we will determine the number of pairs of rabbits that are not yet productive and productive at any point in time.

We will also generalize Fibonacci sequences in the perspective of population dynamics and provide interpretations related to the model being built. We offer a brief exploration of Fibonacci sequences in the form of matrix equations. In this study we will determine the number of rabbit populations in each age group.

2. Research Methodology

This research was carried out with a literature study. Several articles discussing Fibonacci sequences become the foundation in finding the results obtained in this study. In this study two main theorems were obtained. To arrive at the two theorems the following research steps are taken:

Stage 1: Population Dynamics
At this stage, the population growth of rabbits is explained which follows the Fibonacci sequence pattern.

Stage 2: Binet Formula
The number of pairs of rabbits at the time $n = 1, 2$ can be calculated with Binet's Fomula in Equation (1)

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \tag{1}$$

Stage 3: Population Growth with Three Phases
Build a model that produces rabbit population growth with three growth phases (age group). In this step, we use a growth (projection) matrix A with order 3×3

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Stage 4: Number of Rabbits Pairs
Compile and prove a theorem that states the number of pairs of rabbits for each age group.

Stage 5: Generalization of Growth Matrix
Generalizing growth matrices for Fibonacci numbers in perspective of population dynamics and providing interpretations related to the model being built.

Stage 6: Demography Bonus
Explain rabbit population growth with demographic bonus.

3. Discussion Results

Fibonacci sequences are built by terms 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …. This number can be used to analyze population growth. In the study of Mathematics of Demographic, population growth is expressed by equation $P(n) = P(0) + B - D + I - E$; $n = 1, 2, 3, ...$ where $P(0)$ stating the number of populations at the time $n = 0$, and B, D, I and E successively states the number of births, deaths, immigration and emigration, each of which is calculated at intervals $[0, n]$.

Population growth based on Fibonacci sequences is a closed system. In this closed system, there is no population migration and it is assumed there are no dead populations. Thus, the dynamics of population growth are only measured based on the birth rate or population growth. Furthermore, the population growth equation for a closed system is stated by $P(n) = P(0) + B$. In the model, only birth rates, growth rates and survival rates are

recorded on the variable B. Growth rates are further divided into growth rates from infants to juvenils, from juvenils to adults and birth rates. We will build a growth matrix which includes all of these rates.

3.1 Review Fibonacci Sequencess as Population Dynamics.

The numbers on the Fibonacci sequence represent the number of pairs of rabbits in a population system at the time $n = 1,2,3,....$ At the beginning of time ($n = 1$),in the system there is only 1 pair of rabbits, male and female. At a later time ($n = 2$),in the system there are no additional pairs of rabbits. This phenomenon is described as follows.

Our population model starts with a pair of baby rabbits. That is, at the time $n = 1$ there is only 1 pair of rabbits. This rabbit will grow into a juvenil the next month. That is, at the time $n = 2$ still only 1 pair of rabbits. In quantity, the number of pairs of rabbits is still the same. The difference is in the quality or age of the pairs of rabbits, namely the baby at the time $n = 1$ and juvenil at the time $n = 2$. Juvenil rabbits will grow into adult rabbits the following month. An adult characteristic is the pair of rabbits already has offspring who are also a pair of baby rabbits. So, at the time $n = 3$ we have a population 2 pairs of rabbits consisting of 1 pair of adult rabbits and 1 pair of baby rabbits. Next month ($n = 4$),a pair of adult rabbits give birth to another pair of rabbits, and a pair of baby rabbits grow into a pair of juvenil rabbits. So, at the time $n = 4$ we have a population with three age groups, namely infants, juvenils, and adults (Tabel 1). And so on so that we have a population of pairs of rabbits following the Fibonacci number sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …. The development of the number of rabbits is presented in Table 1.

Table 1. Development of Rabbit Population Based on Fibonacci Numbers

Time n	Total Rabbit (couple/s)	Number of Baby Rabbits (couple/s)	Number of Juvenil Rabbits (couple/s)	Number of Adult Rabbits (couple/s)
1	1	1	0	0
2	1	0	1	0
3	2	1	0	1
4	3	1	1	1
5	5	2	1	2
.	.	.	.	.
.	.	.	.	.

Mathematically, Fibonacci sequences begin with two numbers as the next generator of numbers. The two numbers are 1 and 1. The next numbers are the sum of the two previous numbers. Thus, after 1 and 1 is (1 + 1) = 2. Now we have the order of 1, 1, 2. Next is (1 + 2) = 3. And so on. We get 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

3.2 Binet Formula for Fibonacci Sequences

In this article, we define the Fibonacci sequences by order 1, 1, 2, 3, 5, 8, 13, 21, 34, … The numbers on the Fibonacci sequence show the ascending pattern, except for the first two terms. The numbers in the Fibonacci sequence symbolize with $F_1, F_2, F_3, ...$ or F_i, $i = 1, 2, ..., n, ...$ Some researchers write sequences of Fibonacci numbers with 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ….. or 1, 2, 3, 5, 8, 13, 21, 34, ….. For the three types of Fibonacci numbers, 3rd, 4th, ... terms obtained as a sum of the previous two terms.

Each number on the Fibonacci sequence represents the number of pairs of rabbits (male and female) at a certain time. For example, the number 8 means that at any given time there are 8 pairs of rabbits or 16 rabbits each of 8 males and 8 females. At all times, the number of male and female rabbits is always the same. To determine the number of pairs of rabbits at a time n we can use Equation (1). For example, using Equation (1) for $n = 1, 2$ and 4 was obtained

$$F_1 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right) - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right) = \frac{1}{\sqrt{5}}\left(\sqrt{5}\right) = 1$$

$$F_2 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{1}{\sqrt{5}}\left(\frac{2\sqrt{5}}{4} + \frac{2\sqrt{5}}{4}\right) = 1$$

$$F_4 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^4 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^4 = \frac{1}{\sqrt{5}}\left(\left(\frac{6+2\sqrt{5}}{4}\right)^2 - \left(\frac{6-2\sqrt{5}}{4}\right)^2\right)$$

$$= \frac{1}{\sqrt{5}}\left(\frac{24\sqrt{5}}{16} + \frac{24\sqrt{5}}{16}\right) = \frac{1}{\sqrt{5}} \cdot \frac{48}{16}\sqrt{5} = 3$$

As explained in the introduction, we divide the population growth model into two models, the model with two phases and three phases. In the three-phase model we have three age groups for Fibonacci rabbits, namely the baby age group, the juvenil age group and the adult age group. In the two-phase model, we have unproductive groups and productive groups. Productive groups are a combination of adolescent and adult age groups in the three-phase model.

In the three-phase model, we will determine the number of rabbit pairs for the age group of infants, juvenils and adults at each time point. Whereas in the two-phase model, we will determine the number of rabbit pairs in the age group not yet productive and productive.

3.3 Model Pertumbuhan Tiga Fase

Fibonacci numbers can be used to study models of population growth that always increase over time. In this model, we start with a pair of rabbits. If we pay attention, the human population also begins with a pair of people created by Alloh SWT, namely Adam and Eve. Both of them also always gave birth to a pair of twins, male and female. However, the process of increasing human population is not the same as the process of increasing Fibonacci rabbits. The last descendants of Adam and Eve were single, not twin.

Population development that follows Fibonacci numbers runs with the following restrictions:

1. starting with a pair of rabbits (male and female) who are still babies;
2. each pair of rabbits will grow to the phase of baby, juvenil, and adult;
3. transfer from one phase to the next is 1 unit of time of equal length, for example 1 month;
4. if the couple has reached the adult phase (3 months), then the couple will have offspring;
5. offspring for each pair of rabbits is one pair of baby rabbits;
6. pair of rabbits that have reached the adult phase will always give birth every month;
7. no pair of rabbits died, meaning that the mortality rate was 0;
8. no pair of rabbits comes out of the system; and
9. no new rabbit pairs added to the system.

From these rules, a pair of rabbits will have offspring in the third time unit. In the fourth time unit, the pair has 2 pairs of offspring. Thus, in units of time i, the number of pairs of offspring is $i-2$..

The numbers on Fibonacci sequences indicate the number of rabbit pairs with a pair of rabbits meaning male and female. It is assumed at the beginning of the experiment (the first month) we have a pair of rabbits who are still babies. In the second month, the couple grew to become juvenils. In the third month, the rabbit has a pair of children (male and female).

Each pair of rabbits will grow in the same pattern, namely infants, juvenils and adults. Each phase is passed in the same time interval, for example one month. By the time the couple becomes adults, each month will give birth to a pair of children. That is, a pair of rabbits that have entered the adult phase will give birth continuously every month. So, the baby bunny pair will give birth to a pair of children in the third, fourth, fifth month and so on. Thus, we have a population of Fibonacci rabbits consisting of three age groups according to the phases of rabbit growth, namely babies, juveniles and adults.

Table 2. Three-phase Fibonacci Rabbit Growth Models for Each Age Group

Kelompok Usia	Waktu (n)								
	1	2	3	4	5	6	7	8	9
Babies	A	0	B^A	C^A	D^A E^B	F^A G^B H^C	I^A J^B K^C L^D M^E	N^A O^B P^C Q^D R^E S^F T^G U^H	V^A W^B X^C Y^D Z^E α^F β^G γ^H δ^I ε^J λ^K ρ^L σ^M
Juveniles	0	A	0	B^A	C^A	D^A E^B	F^A G^B H^C	I^A J^B K^C L^D M^E	N^A O^B P^C Q^D R^E S^F T^G U^H
Adults	0	0	A	A	A B^A	A B^A C^A	A B^A C^A D^A E^B	A B^A C^A D^A E^B F^A G^B H^C	A B^A C^A D^A E^B F^A G^B H^C I^A J^B K^C L^D M^E
Total	1	1	2	3	5	8	13	21	34

The Fibonacci rabbit population development model is a closed model, meaning that no rabbits enter and exit the system (no migration). The mortality rate for each pair of rabbits is 0, meaning that no rabbit pair has died. Thus, over time the rabbit population will always increase. Fibonacci rabbit growth models for each age group are given in Table 2 and Table 3.

In Table 2, the population begins with a pair of baby rabbits A. In the second month, the pair of rabbits A grows into juveniles. In the third month, the pair of rabbits A grows into adulthood and has a pair of children, namely B^A with the symbol B^A means the pair of rabbits B is descended from the pair of rabbits A. The number 0 in Table 2 states there is no population.

The development of rabbit populations in Table 2 was subsequently converted in Table 3 to express the number of pairs of rabbits in each age group over time. Based on Table 3, in the first month the number of pairs of rabbits for the age group of infants, adolescents and adults respectively are 1, 0, and 0. In the fifth month, the number of pairs of rabbits for the age group of infants, adolescents and adults are 2, 1 and 2, respectively. The number of pairs of baby, juvenile and adult rabbits for successive time points results in rows of Fibonacci numbers.

From Table 2, the number of pairs of babies and adults for each time point $n \geq 3$ is the same. For example, for the 9th month, the number of pairs of baby and adult rabbits is the same, namely 13. Because the Fibonacci number for the 9th term is 34 (calculated by equation (1)), the number of pairs of juvenile rabbits is. In general, we have a pair of teenage rabbits at the time i is $J_i = F_{i+2} - (2 \times F_i)$, where F_i expresses the Fibonacci number i-th, where $i = 6, 7, 8, ...$, $J_4 = 1$ and $J_5 = 1$. Fibonacci sequences in successive baby, juvenil and adult age groups start for $i = 3$, $i = 4$ and $i = 3$ (see Table 3)

Table 3. Number of Pairs of Three-phase Fibonacci Rabbit Growth Models for Each Age Group

Kelompok Usia	Waktu (i)											
	1	2	3	4	5	6	7	8	9	10	11	12
Babies	1	0	1	1	2	3	5	8	13	21	34	55
Juveniles	0	1	0	1	1	2	3	5	8	13	21	34
Adults	0	0	1	1	2	3	5	8	13	21	34	55
Total	1	1	2	3	5	8	13	21	34	55	89	144

In Table 3, we get a line of Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, ……. (see total row). If we pay attention, it turns out that each age group of infants, juveniless and adults also grows by forming a Fibonacci number sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, ……. That is, each age group increases following the Fibonacci number sequence. For baby and adult age groups, the Fibonacci number sequence starts from 3th months. For juvenile age groups starting from the 4th month.

For example B_i, J_i, A_i successively states the number of pairs of baby, juvenile and adult rabbits at a time i. Let F_i is the Fibonacci number i-th. Because of the many pairs of baby rabbits, teenagers and adults in the form of Fibonacci numbers, then respectively B_i, J_i, A_i have a relationship with F_i, that is

$$B_i = F_{i-2} \quad ; i = 3, 4, ... \text{ where } B_1 = 1 \text{ and } B_2 = 0 \tag{2}$$

$$R_i = F_{i-3} \quad ; i = 4, 5, ... \text{ where } R_1 = 0,\ R_2 = 1 \text{ and } R_3 = 0 \tag{3}$$

$$A_i = F_{i-2} \quad ; i = 3, 4, ... \text{ where } A_1 = 0 \text{ and } A_2 = 0. \tag{4}$$

Equation (2), (3), and (4) states that Fibonacci numbers in the infant and adult age groups are two steps late, whereas in the juvenil group three steps late. Next, using Equation (2), we can find out the number of pairs of rabbits in the baby age group at the time $n = 12$ month, that is $B_{12} = F_{10}$ where F_{10} calculated using Equation (1).

3.4 Number of Rabbit Pairs for Each Growth Phase (Age Group)

Binet's formula is used to determine the number of rabbit pairs at a certain time. In this section a theorem is constructed that can be used to determine the number of pairs of rabbits in each age group. Suppose the Fibonacci sequences obtained from Table 3 are written 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …. This squence is applies to babies B, juveniles J, and addults A. From Table 3 we get a relationship

$$F_{i+3} = B_{i+1} + J_i + A_{i+1} \quad ; i = 1, 2, 3, ... \tag{5}$$

Equation (5) explains that a Fibonacci number $F_4 = 8$ obtained from the sum of Fibonacci numbers $B_4 = 3$, $R_3 = 2$ and $A_4 = 3$.

Next, we will declare a growth vector consisting of three lines. The first, second and third row in a row states the number of pairs of babies, juveniles and adults for each time point $i = 1, 2, 3, ...$ For $i = 1$ we call the

known initial population. We will predict the number of pairs of rabbits in each age group for the years to come. To get an estimate of the number of rabbits, we make an equation

$$P(i+1) = A \cdot P(i) \qquad ;\ i = 1, 2, 3, \ldots.$$

where A projection matrices are square-sized matrices 3×3 and $P(i)$ is a vector of growth in time i. The development of rabbits is expressed in three age groups so that the vector $P(i)$ has three rows, which respectively represent the number of pairs of baby, juvenile and adult rabbits. Following are some growth vectors obtained from Table 3.

$$P(1) = \begin{pmatrix} P_1(1) \\ P_2(1) \\ P_3(1) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad P(2) = \begin{pmatrix} P_1(2) \\ P_2(2) \\ P_3(2) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad P(3) = \begin{pmatrix} P_1(3) \\ P_2(3) \\ P_3(3) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Using Equation (2) we obtain the following equations, with a projection matrix A by order 3×3.

$$P(2) = A \cdot P(1) \qquad\qquad P(3) = A \cdot P(2)$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$P(4) = A \cdot P(3) \qquad\qquad P(5) = A \cdot P(4)$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Theorem 1: *Let A is a projection matrix for the three-phase Fibonacci rabbit growth model with* $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. *The population of Fibonacci rabbits at the time $i+1$ is* $P(i+1) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot P(i)$; $i = 1, 2, 3, \ldots.$

Using Theorem 1, we can predict the number of pairs of rabbits in the future. For example $i = 2$, obtained by the number of pairs of rabbits 3 months later

$$P(3) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot P(2) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

In the same way, we get a pattern $A \cdot P(1) = P(2) \rightarrow A \cdot P(2) = P(3) \rightarrow A \cdot P(3) = P(4)$ and so on. The pattern is expressed as follows.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix} \rightarrow \ldots.$$

Corollary 1: *Let* $P(i) = P\begin{pmatrix} P_1(i) \\ P_2(i) \\ P_3(i) \end{pmatrix}$ *where $P_1(i)$, $P_2(i)$, and $P_3(i)$ simultaneously contributing to age partners in infants, juveniles and adults at the time i. Number of rabbit pairs in the age group of infants, juveniles and adults at the time $i+1$ consecutively is $P_1(i+1) = P_2(i) + P_3(i)$, $P_2(i+1) = P_1(i)$, and $P_3(i+1) = P_2(i) + P_3(i)$.*

Suppose we want to know the number of rabbit pairs in the baby age group in the next five years. We can use Theorem 1 and Corollary 1. With theorem 1 we get the results

$$P(6)=\begin{pmatrix}0&1&1\\1&0&0\\0&1&1\end{pmatrix}\cdot P(5)=\begin{pmatrix}0&1&1\\1&0&0\\0&1&1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\2\end{pmatrix}=\begin{pmatrix}3\\2\\3\end{pmatrix}.$$

The number of pairs of rabbits in the infant age group is the first row of vectors $P(6)$, i.e. 3. If we use Corollary 1, the number of pairs of rabbits in the baby's age group is

$P_1(n+1)=P_2(n)+P_3(n)\Leftrightarrow P_1(6)=P_2(5)+P_3(5)=1+2=3$

Theorem 1 and Corollary 1 are identical. Both are used if we know for sure the population a month before. Both are used if we know for sure the population a month before $i=1$ and will search for populations n next month then we can modify Theorem 3:

$$\begin{aligned}P(i+n)&=A\cdot P(i+(n-1))=A\cdot P(i-1+n)\\&=A\cdot(A\cdot P(i-2+n))=A\cdot A\cdot P(i-2+n)=A^2\cdot P(i-2+n)\\&=A^3\cdot P(i-3+n)\\&=.....\\&=A^i\cdot P(i-n+n)=A^n\cdot P(i)\end{aligned}$$

Thus, $P(i+n)=A^n\cdot P(i)$. If we take $i=1$, then obtained $P(1+n)=A^n\cdot P(1)$ or

$$P(1+n)=\begin{pmatrix}0&1&1\\1&0&0\\0&1&1\end{pmatrix}^n\cdot P(1) \tag{6}$$

where $P(1)=\begin{pmatrix}1\\0\\0\end{pmatrix}$

Now we will examine the relationship between the projection matrix A with population vectors $P(1)$, $P(2)$, $P(3)$ and so on. From Table 2 we get

$P(1)=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $P(2)=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $P(3)=\begin{pmatrix}1\\0\\1\end{pmatrix}$, $P(4)=\begin{pmatrix}1\\1\\1\end{pmatrix}$, $P(5)=\begin{pmatrix}2\\1\\2\end{pmatrix}$, $P(6)=\begin{pmatrix}3\\2\\3\end{pmatrix}$, $P(7)=\begin{pmatrix}5\\3\\5\end{pmatrix}$, …. If the projection matrix A we multiply by itself, matrices are obtained A^2, A^3, A^4,.... where

$$A=\begin{pmatrix}0&1&1\\1&0&0\\0&1&1\end{pmatrix},\ A^2=\begin{pmatrix}1&1&1\\0&1&1\\1&1&1\end{pmatrix},\ A^3=\begin{pmatrix}1&2&2\\1&1&1\\1&2&2\end{pmatrix},\ A^4=\begin{pmatrix}2&3&3\\1&2&2\\2&3&3\end{pmatrix},\ A^5=\begin{pmatrix}3&5&5\\2&3&3\\3&5&5\end{pmatrix},$$

Note that the first column in the matrix A, A^2, A^3, A^4,.... successively are population vectors $P(2)$, $P(3)$, $P(4)$, $P(5)$, $P(6)$, …. So, the first column matrix A^n is a population vector $P(1+n)$, where $n=1,2,3,....$ Based on Equation (6), that is $P(1)\cdot A^n=P(1+n)$, then the solution for Equation (6) is the first column of the matrix A^n.

Teorema 2: *The population vector for the coming month is the first column of the matrix* A^n *where* $A=\begin{pmatrix}0&1&1\\1&0&0\\0&1&1\end{pmatrix}$ *and* $n=1,2,3,....$

Proof of Theorem 1, 2 and Corollary 1 submitted to the reader.

3.5 Generalization of Growth Matrix

Projection (growth) matrix A in Equation (2) we can generalize by giving values in the form of real numbers between 0 and 1 to the matrix elements. We have a generalized projection matrix A_m in Equation (7)

$$A_m = \begin{pmatrix} 0 & \eta & \delta \\ \lambda & 0 & 0 \\ 0 & \rho & \tau \end{pmatrix} \tag{7}$$

The parameters in the matrix A_m can be interpreted as follows. Parameter η states the survival rate of juveniles who reach the adult group. Parameter δ states the survival rate of the adult group to continue to live and produce new offspring. Parameter λ states the survival rate of the group of infants reaching the juvenil group. Parameter ρ states the survival rate of juveniles who reach the adult group. The last, parameter τ states the survival rate of the adult group to continue to live and produce new offspring.

3.5 Demographic Bonuses in the Fibonacci Rabbit Population Perspective

In the three-phase growth model, we can consider the age group of infants and adults as non-productive age groups. This group will be borne by the juvenil age group. The number of age groups for infants and adults are the same. For example, in Table 3, for $i = 8$, the number of baby and adult rabbits is 8, and the number of juvenile rabbits is 5 so for $i = 8$ the total number of rabbits is 21. The ratio of non-productive age groups to productive age groups is 16/5 = 3.2. That is, 1 individual bears 3.2 unproductive individuals. Table 6 shows the ratio between non-productive age groups and productive age groups.

Table 6. Ratios between Unproductive and Productive Populations in the Three-phase Growth Model

Individual Group	Time (i)											
	1	2	3	4	5	6	7	8	9	10	11	12
Unproductive	1	0	2	2	4	6	10	16	26	42	68	110
Productive	0	1	0	1	1	2	3	5	8	13	21	34
Ratio	-	0	-	2	4	3	3.33	3.20	3.25	3.23	3.24	3.24

Individual Group	Time (i)									
	13	14	15	16	17	18	19	20	21	22
Unproductive	178	288	466	754	1220	1974	3194	5168	8362	13530
Productive	55	89	144	233	377	610	987	1597	2584	4181
Ratio	3.24	3.24	3.24	3.24	3.24	3.24	3.24	3.24	3.24	3.24

From these observations it was concluded that

$$\lim_{i \to \infty} \frac{B_i + A_i}{J_i} = 3.24$$

This figure means that every 100 productive individuals bears 324 nonproductive individuals. In the case of a demographic bonus, 100 productive individuals will bear 40-50 non-productive individuals. Non-productive individuals are non-productive individuals (the infant age group) and non-productive groups (the adult age group). So, the hypothetical population based on Fibonacci numbers is very far from being expected to get a demographic bonus.

We return to the three-phase growth model. Mathematically, a demographic bonus can be achieved if $\frac{40}{100} \le \frac{B_i + A_i}{J_i} \le \frac{50}{100}$. We can choose parameters that satisfy the inequality. For example,

$$\frac{40}{100} \le \frac{B_i + A_i}{J_i} \le \frac{50}{100}$$

$$\Leftrightarrow \frac{40}{100} \le \frac{P_1(n+1) + P_3(n+1)}{P_2(n+1)} \le \frac{50}{100}$$

$$\Leftrightarrow \frac{40}{100} \le \frac{(\eta + \rho) \cdot P_1(n) + (\delta + \tau) \cdot P_3(n)}{\lambda \cdot P_2(n)} \le \frac{50}{100}$$

Let $(\eta + \rho) = (\delta + \tau) = \theta$

$$\Leftrightarrow \frac{40}{100} \le \frac{\theta \cdot (P_1(n) + P_3(n))}{\lambda \cdot P_2(n)} \le \frac{50}{100}$$

$$\Leftrightarrow \frac{40}{100} \le \frac{\theta}{\lambda} \frac{P_1(n) + P_3(n)}{P_2(n)} \le \frac{50}{100}$$

$$\Leftrightarrow \frac{40}{100} \le 3.24 \cdot \frac{\theta}{\lambda} \le \frac{50}{100}$$

$$\Leftrightarrow \frac{6}{100} \le \frac{\theta}{\lambda} \le \frac{7{,}5}{100}$$

Choose $\lambda = 1$

$$\Leftrightarrow \frac{6}{100} \le \theta \le \frac{7{,}5}{100}$$

$$\Leftrightarrow 0{,}060 \le \theta \le 0{,}075$$

We take the example $\theta = 0.070$ for the projection matrix $A_m = \begin{pmatrix} 0 & \eta & \delta \\ \lambda & 0 & 0 \\ 0 & \rho & \tau \end{pmatrix} = \begin{pmatrix} 0 & 0.035 & 0.035 \\ 1 & 0 & 0 \\ 0 & 0.035 & 0.035 \end{pmatrix}$. Number of pairs of baby and adult rabbits at the time $n = 9$ is the same ie $(\eta + \rho) \cdot P_1(n) = (\delta + \tau) \cdot P_3(n) = (0.07 + 0.07) \cdot 13 = 91/50$. As a result, the number of baby and adult rabbits is 91/25. While the number of pairs of teenage rabbits is $\lambda \cdot P_2(n) = 1 \cdot 8 = 8$. Dependency ratio is ((91/25)/8) = 91/200 = 45.5/100. That is, 45.5 non-productive individuals are covered by 100 productive individuals. This example shows that modifying the projection matrix in a 3-phase growth model can produce demographic bonuses.

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