Optimal Reinsurance and Investment Problem Under Fractional Power Utility Function

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Abstract

This paper discusses the optimal problem of reinsurance and investment for insurance companies with a fractional power utility function. Insurance companies can buy reinsurance contracts and invest their wealth in risk-free or risk-free financial securities. It is assumed that the insurance company surplus process is estimated using Brownian motion. The aim of the insurance company is to seek optimal reinsurance and investment strategies by maximizing expected utility expectations from the final wealth. The explicit form for the optimal strategy is determined by the stochastic optimal control theory approach, which uses the Hamilton Jacobi Bellman equations.

Keywords:
Reinsurance, Investment, Fractional power utility, optimal control theory, Hamilton Jacobi Bellman equations.

1. Introduction

Human life is inseparable from the loss of life or material that can come at any time. The uncertainty that can cause damages is referred to as risk. One of human's efforts to overcome the risk is by transferring the risk with other parties. How to transfer the risk of the other party needs an agreement. Such an agreement is called an insurance agreement. Insurance is an agreement between two or more parties where the insured party binds itself to the guarantor with an agreement. The insured party has an obligation to pay a number of contributions to the guarantor, which is referred to as a premium. At the same time, the guarantor has an obligation to provide compensation if the insured person suffers a loss of risk that occurs at a time. Insurance contracts are designed to meet specific needs and therefore have many features not found in many other types of contracts.

On the other hand, insurance companies (insurers) are also likely to incur losses if there is a very large claim at one time, which means the insurance company also needs to transfer the risk to the other party. Other parties who are responsible for some of the risks of insurance companies are referred to as reinsurance companies. Reinsurance is an agreement or method by which an insurance company surrenders all or part of the risk to another insurer known as a reinsurer. Reinsurance functions as a place to spread risk, thus providing protection to the insurer from loss.
Premiums that enter as the insurance company's cash flow must be managed as well as possible so that all company obligations can be fulfilled on time without disrupting the company's financial stability. One way to manage the finances that enter the insurance company is to conduct investment activities. Investment is an important activity for insurance companies. Companies must be able to demonstrate reliable underwriting capabilities and maintain the level of corporate financial solvency. The investment needs to be done because the premiums that go into the company's cash as revenue are strived to produce good returns to maintain the company's financial stability in the future. In general, insurance companies will invest their funds in several investment instruments, such as time deposits, stocks, corporate bonds, government bonds, and so forth.

In investing, insurance companies must correctly choose and allocate a portion of their finances for investment because investment instruments have different characters. Therefore, insurance companies must have an optimal strategy to minimize bankruptcy opportunities. In economics, a utility function is used to measure the wealth of an insurance company and the total risk that a company is able to bear in the hope of increasing wealth. By using the utility function approach, it is expected that the optimal strategy can be determined by maximizing the utility's expectations (Cao, Y). Some researchers have discussed the problem of optimization investment and insurance with various models and various utility functions. Gu, A. et al. investigate insurance and investment issues for insurance companies devoted to excess loss reinsurance and the CEV model for the price of risk assets. Li, Q., Gu, M., and Wang, Y. et al. also uses the CEV model on the price of risk assets in determining the optimal strategy investment and reinsurance. Chunxiang, A., Li, Z. discussed the problem of investment and insurance of excess of loss using the delay under Heston's SV model. Hu, X. et al. get optimal retention for stop-loss reinsurance with incomplete information. Chunxiang, A. et al. investigate the optimal investment strategy for insurance companies and excess of loss reinsurance companies using the delay under the jump-diffusion risk process and the CEV model. Hu, H. et al. use the Jump Diffusion Process in determining optimal investment and reinsurance strategy. Li, D. et al. investigated the optimal investment problem for an insurer and a reinsurer under the proportional reinsurance with an exponential utility function. Sheng, D. discussed an explicit solution of the optimal reinsurance investment problem with the promotion budget under a power utility function. Li, D. studied the optimal reinsurance investment problem for maximizing the product of the insurer's and reinsurer's utilities under a CEV model. Lhedioha, S.A. et al. considered the optimal portfolios of an insurer and a reinsurer under proportional reinsurance and power utility preference.

From the above literature, many papers use the exponential utility function. Specifically, in this paper, we choose a fractional power utility function, and we find a strategy that is maximizing the expected fractional power utility function with the constraint stochastic differential equation. In our model, the claim process is assumed based on Brownian motion with drift. The insurer is allowed to invest a risky asset followed Black Scholes Equation and a risk-free asset in the financial market. First, we derive the model of the insurer wealth process in a stochastic differential equation. Second, we find the explicit solution for the problem with Hamiltonian Jacobi Bellman. In section 2, we introduce the mathematical modeling of the problem. Section 3 provides the optimal proportional reinsurance and investment strategies under the fractional power utility of maximizing the expected utility function. In section 4 concludes this paper.

2. Mathematical Modelling of the Problem

In this section, a model of the wealth process is modeled. However, previously studied first in a row about some notations.

Below are given some notations needed in the formation of the model.

- x_0: initial assets.

- c: the premium rate of the insurer.
- c_1: the constant rate premium paid an insurer to a reinsurer.
- θ, η: the safe loading of the insurer and reinsurer, respectively.
- $\pi(t)$: the amount of the wealth invested in the risky asset at time t by the insurer.
- $p(t)$: the proportion reinsured at time t.
- $\{X(t), t \geq 0\}$: the surplus process of the insurer.

In this paper, the claim process is written as $C(t)$ and defined based on Brownian motion with drift as

$$dC(t) = a\,dt - b dW_0(t) \tag{1}$$

with a, b are positive constants and $W_0(t)$ a standard Brownian motion. Suppose the premium is paid continuously at a constant rate following the model $c = (1+\theta)a$ with $\theta > 0$. The reserve of the insurer at time t defined with following Cramer Lundberg Model as

$$R(t) = x_0 + c\,t - C(t)$$

and from (1) we have a surplus process for the insurer as

$$dR(t) = c\,dt - dC(t) = c\,dt - (a\,dt - bdW_0(t)) = a\,\theta\,dt + bdW_0(t)$$

Furthermore, we assume that the insurer can buy a reinsurance contract to reduce risk. Suppose $p(t)$ represents the proportion for reinsurance at time t, and $c_1 = (1+\eta)a$ is a constant rate premium paid an insurer to a reinsurer with $\eta > 0$. So that it considers the reinsurance, then the surplus process for the insurer is defined as

$$\begin{aligned} dR_1(t) &= c\,dt - \big(1-p(t)\big)dC(t) - c_1\,p(t)dt \\ &= cdt - \big(1-p(t)\big)\big(a\,dt - bdW_0(t)\big) - (1+\eta)a\,p(t)dt \\ &= \big(\theta - \eta p(t)\big)a\,dt + b(1-p(t))dW_0(t) \end{aligned} \tag{2}$$

In addition to buying a reinsurance contract, the insurer can also invest part of their wealth in the financial market. The investment is in a free risk asset and a risk asset. Risk-free asset prices $S_0(t)$ follow the equation

$$dS_0(t) = r\,S_0(t)dt \tag{3}$$

with $r > 0$ is the interest rate for free risk assets. Then the price of the risky asset is described by the Black Scholes model

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \tag{4}$$

with $\mu > r$ is the interest rate for risk asset and $W(t)$ a standard Brownian motion. Suppose $\pi(t)$ is the amount of wealth of insurer invested in risk assets at time t and the remainder $X(t) - \pi(t)$ invested in risk free assets. Based on the explanation above, the investment and reinsurance strategies formed are a pair $\beta = \big(\pi(t), p(t)\big)$.

Strategy β admissible if $0 \leq p(t) \leq 1$ and $E\left[\int_0^T \pi^2(t)dt\right] < \infty$ with $t \in [0, T]$. Relating to strategy β, the model of insurer wealth process development to follow stochastic differential equation as

$$dX(t) = dR_1(t) + \pi(t)\frac{dS(t)}{dt} + (X(t) - \pi(t))\frac{dS_0(t)}{dt}. \tag{5}$$

From (2), (3), (4) and (5), we can write the wealth process of the insurer as

$$\begin{aligned} dX(t) &= \big(\theta - \eta p(t)\big)a\,dt + b\big(1-p(t)\big)dW_0(t) + \pi(t)\big(\mu\,dt + \sigma\,dW(t)\big) + \big(X(t) - \pi(t)\big)rdt \\ &= \big[rX(t) + \pi(t)(\mu - r) + \big(\theta - \eta p(t)\big)a\big]dt + b\big(1-p(t)\big)dW_0(t) + \pi(t)\sigma\,dW(t). \end{aligned} \tag{6}$$

3. Optimal Solution

In this section, we try to find a strategy $(\pi(t), p(t))$ that maximizes the expected utility function. Suppose that the insurer has a utility function with form fractional power utility function which is given by

$$U(x) = x^{\alpha} \tag{7}$$

where $0 < \alpha < 1$, the control optimal stochastic problems in this paper is to maximize objective function:

$$M(t,x) = \max_{(\pi(t),p(t))} E[U(X(T))]$$

with constraint (6).
The theorem that follows gives that optimization of the insurer wealth.

Theorem 1. If W and W_0 are independent ($cov\ [W, W_0] = 0$) and the insurer has fractional power utility function (7), then the optimal strategy maximizes the expected utility function

at terminal time T is to invest at each time $t \leq T$ is

$$\pi^* = \frac{(\mu - r)x}{\sigma^2(1-\alpha)}$$

with optimal proportion reinsurance

$$p^* = 1 + \frac{\eta a x}{b^2(\alpha - 1)}$$

and value function

$$M(t,x) = x^{\alpha}\, K \exp\left(\left(r\,\alpha + \frac{\theta\, a}{x} - \frac{\eta\, a\, \alpha}{x} - \frac{1}{2}\frac{(\mu-r)^2\alpha}{\sigma^2(\alpha-1)} - \frac{1}{2}\frac{\eta^2 a^2\, \alpha}{b^2(\alpha-1)}\right)t\right)$$

where $K > 0, a > 0, b > 0, \theta > 0, \eta > 0, \mu > 0$ and $\sigma > 0$ are constants.

Proof. To get the optimal control variable solution β, we will use the Hamiltonian Jacobi Bellman equation (HJB). The HJB equation is often used to solve optimal stochastic control problems, aiming to determine the control variable $\beta^* = \left(\pi^*(t), p^*(t)\right)$ with constraints in the form stochastic differential equation.
Because W and W_0 are independent ($cov\ [W, W_0] = 0$), so the corresponding HJB equation is

$$M_t + \sup_{\beta}\left\{[rx + \pi(\mu - r) + (\theta - \eta p)a]M_x + \tfrac{1}{2}[\pi^2\sigma^2 + b^2(1-p)^2]M_{xx}\right\} = 0 \tag{8}$$

with boundary condition $M(T,x) = U(x)$, where M_t, M_x, M_{xx} denote partial derivatives of first and second orders with respect to t and x.
Differentiating (8) with respect to $\pi(t)$, we have

$$(\mu - r)M_x + (\pi\sigma^2)M_{xx} = 0$$

with simplify, we get

$$\pi^*(t) = -\frac{\mu - r}{\sigma^2}\frac{M_x}{M_{xx}}.$$

Substituting $\pi^*(t)$ into HJB equation (8), we obtain

$$M_t + rxM_x + (\theta - \eta p)a\, M_x + (\mu - r)\left(-\frac{\mu - r}{\sigma^2}\frac{M_x}{M_{xx}}\right)M_x + \frac{1}{2}b^2(1-p)^2 M_{xx} + \frac{1}{2}\sigma^2\left[\frac{(\mu-r)^2}{\sigma^4}\frac{M_x^2}{M_{xx}^2}\right]M_{xx} = 0$$

$$\Leftrightarrow M_t + rxM_x + (\theta - \eta p)a\, M_x - \frac{(\mu-r)^2}{\sigma^2}\frac{M_x^2}{M_{xx}} + \frac{1}{2}b^2(1-p)^2 M_{xx} + \frac{1}{2}\sigma^2\left[\frac{(\mu-r)^2}{\sigma^4}\frac{M_x^2}{M_{xx}}\right] = 0$$

$$\Leftrightarrow M_t + rxM_x + (\theta - \eta p)a\, M_x - \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\frac{M_x^2}{M_{xx}} + \frac{1}{2}b^2(1-p)^2 M_{xx} = 0.$$

(9)

with $M(T,x) = U(x) = x^{\alpha}$. Then, differentiating with respect to $p(t)$ in (9) and we have

$$-\eta a M_x - b^2(1-p)M_{xx} = 0$$
$$\Leftrightarrow b^2 p M_{xx} = \eta a\, M_x + b^2\, M_{xx}$$
$$\Leftrightarrow p^* = 1 + \frac{\eta a M_x}{b^2 M_{xx}}.$$

According to the fractional power utility function described by equation (7), we try to find the solution (9) in the following way

$$M(t,x) = x^{\alpha}\, f(t)$$

then

$$M_x = \alpha x^{\alpha-1} f(t)$$
$$M_{xx} = \alpha(\alpha-1)x^{\alpha-2}\, f(t)$$
$$M_t = x^{\alpha}\, f'(t)$$

Plugging these derivatives M_t , M_x and M_{xx} into the HJB equation (8), we have

$$\begin{aligned}
x^{\alpha} f'(t) + \left[rx - \frac{(\mu-r)^2}{\sigma^2(\alpha-1)} + \theta a - \eta a - \frac{\eta^2\, a^2 x}{b^2(\alpha-1)} \right] \alpha x^{\alpha-1} f(t) \\
+ \frac{1}{2}\left[\left(\frac{(\mu-r)x}{\sigma^2(\alpha-1)} \right)^2 \sigma^2 + b^2 \left(\frac{\eta a\, x}{b^2(\alpha-1)} \right)^2 \right] \alpha(\alpha-1)\, x^{\alpha-2}\, f(t) = 0 \\
\Leftrightarrow x^{\alpha} f'(t) + \left[r\,\alpha\, x^{\alpha} - \frac{(\mu-r)^2 \alpha\, x^{\alpha}}{\sigma^2(\alpha-1)} + \theta\, a x^{\alpha-1} - \eta a \alpha x^{\alpha-1} - \frac{\eta^2 a^2 x^{\alpha}\, \alpha}{b^2(\alpha-1)} \right] f(t) \\
+ \frac{1}{2}\left[\frac{(\mu-r)^2 x^{\alpha}\alpha}{\sigma^2(\alpha-1)} + \frac{\eta^2 a^2\, a x^{\alpha}}{b^2(\alpha-1)} \right] f(t) = 0 \\
\Leftrightarrow x^{\alpha} f'(t) + \left[r\,\alpha\, x^{\alpha} + \theta\, a x^{\alpha-1} - \eta a \alpha x^{\alpha-1} - \frac{1}{2}\frac{(\mu-r)^2 \alpha\, x^{\alpha}}{\sigma^2(\alpha-1)} - \frac{1}{2}\frac{\eta^2 a^2 x^{\alpha}\, \alpha}{b^2(\alpha-1)} \right] f(t) = 0
\end{aligned}$$

$$\Leftrightarrow f'(t) + \left[r\,\alpha + \frac{\theta\, a}{x} - \frac{\eta\, a\, \alpha}{x} - \frac{1}{2}\frac{(\mu-r)^2 \alpha}{\sigma^2(\alpha-1)} - \frac{1}{2}\frac{\eta^2 a^2\, \alpha}{b^2(\alpha-1)} \right] f(t) = 0.$$

The solution of the above differential equation is

$$f(t) = K \exp\left(\left(r\,\alpha + \frac{\theta\, a}{x} - \frac{\eta\, a\, \alpha}{x} - \frac{1}{2}\frac{(\mu-r)^2 \alpha}{\sigma^2(\alpha-1)} - \frac{1}{2}\frac{\eta^2 a^2\, \alpha}{b^2(\alpha-1)} \right) t \right)$$

for $K \geq 0$.

So, we obtain the value function is

$$M(t,x) = x^{\alpha}\, K \exp\left(\left(r\,\alpha + \frac{\theta\, a}{x} - \frac{\eta\, a\, \alpha}{x} - \frac{1}{2}\frac{(\mu-r)^2 \alpha}{\sigma^2(\alpha-1)} - \frac{1}{2}\frac{\eta^2 a^2\, \alpha}{b^2(\alpha-1)} \right) t \right)$$

and the optimal strategy $\beta^* = \left((\pi^*, p^*) \right)$ as follows:

$$\pi^* = \frac{(\mu-r)x}{\sigma^2(1-\alpha)}$$

and

$$p^* = 1 + \frac{\eta a x}{b^2(\alpha-1)}.$$

Theorem 2. If W and W_0 are dependent ($cov\ [W, W_0] = \rho\, t, \rho \in [-1,1]$) and the insurer has fractional power utility function as in (7), then the optimal strategy that maximizes the expected utility function

at terminal time T is to invest at each time $t \le T$ is

$$\pi^* = \frac{\left(-2\left(\rho^2 + \frac{1}{2}\right)(\mu - r)b + a\rho\eta\sigma\right)x}{b(\alpha-1)(1+\rho^2)\sigma^2}$$

with optimal proportion reinsurance

$$p^* = \frac{(\alpha-1)b^2(1+\rho^2)\sigma + \eta x\, a\, \sigma - \rho(\mu - r)xb}{b^2(\alpha-1)(1+\rho^2)\sigma}$$

and value function

$$\begin{aligned} M(t,x) = x^\alpha K_1 \exp\Bigg(\Bigg(\Bigg[& rx + \frac{\left(-2\left(\rho^2 + \frac{1}{2}\right)(\mu - r)b + a\rho\eta\sigma\right)x(\mu - r)}{b(\alpha-1)(1+\rho^2)\sigma^2} \\ & + \left(\frac{-\left((\alpha-1)b^2(1+\rho^2)\sigma + \eta x\, a\, \sigma - \rho(\mu-r)xb\right)\eta}{(\alpha-1)b^2(1+\rho^2)\sigma} + \theta\right) a\Bigg]\frac{\alpha}{x} \\ & + \frac{1}{2}\Bigg[\frac{\left(-2\left(\rho^2+\frac{1}{2}\right)(\mu-r)b + a\rho\eta\sigma\right)^2 x^2}{b^2(\alpha-1)^2(1+\rho^2)\sigma^2} \\ & + b^2\left(1 - \frac{(\alpha-1)b^2(1+\rho^2)\sigma + \eta x\, a\, \sigma - \rho(\mu-r)x\, b}{(\alpha-1)b^2(1+\rho^2)\sigma}\right)^2 \\ & + \frac{2\left(-2\left(\rho^2+\frac{1}{2}\right)(\mu-r)b + a\rho\eta\sigma\right)x\left(1 - \frac{(\alpha-1)b^2(1+\rho^2)\sigma + \eta x a\sigma - \rho(\mu-r)xb}{(\alpha-1)b^2(1+\rho^2)\sigma}\right)\rho}{(\alpha-1)(1+\rho^2)\sigma}\Bigg]\frac{\alpha(\alpha-1)}{x^2}\Bigg)t\Bigg) \end{aligned}$$

where $K_1 > 0, a > 0, b > 0, \theta > 0, \eta > 0, \mu > 0$ and $\sigma > 0$ are constants.

Proof. Because W and W_0 are independent ($cov\ [W, W_0] = \rho\, t, \rho \in [-1,1]$), so the corresponding HJB equation is

$$M_t + \sup_\beta \left\{[rx + \pi(\mu - r) + (\theta - \eta p)a]M_x + \tfrac{1}{2}[\pi^2\sigma^2 + b^2(1-p)^2 + 2\pi\sigma b(1-p)\rho]M_{xx}\right\} = 0 \quad (10)$$

with boundary condition $M(T,x) = U(x)$, where M_t, M_x, M_{xx} denote partial derivatives of first and second orders with respect to t and x.
Differentiating (10) with respect to $\pi(t)$, we have

$$(\mu - r)M_x + (\pi\sigma^2 + \sigma\, b(1-p)\rho)M_{xx} = 0$$

with simplify, we get

$$\pi^*(t) = -\frac{\mu - r}{\sigma^2}\frac{M_x}{M_{xx}} - \frac{b(1-p)\rho}{\sigma}.$$

Substituting $\pi^*(t)$ into HJB equation (8), we obtain

$$\begin{aligned} M_t + rxM_x + (\theta - \eta p)a\, M_x + (\mu - r)\left(-\frac{\mu - r}{\sigma^2}\frac{M_x}{M_{xx}} - \frac{b(1-p)\rho}{\sigma}\right)M_x + \frac{1}{2}b^2(1-p)^2 M_{xx} \\ + \frac{1}{2}\sigma^2\left(\frac{(\mu-r)^2}{\sigma^4}\frac{M_x^2}{M_{xx}^2} + \frac{b^2(1-p)^2\rho^2}{\sigma^2} + \frac{2(\mu-r)b(1-p)\rho}{\sigma^3}\frac{M_x}{M_{xx}}\right)M_{xx} \\ + \sigma b(1-p)\rho\left(-\frac{\mu-r}{\sigma^2}\frac{M_x}{M_{xx}} - \frac{b(1-p)\rho}{\sigma}\right)M_{xx} = 0 \end{aligned}$$

$$\Leftrightarrow M_t + rxM_x + (\theta - \eta p)a\, M_x - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}\frac{M_x^2}{M_{xx}} + \frac{1}{2}b^2(1-p)^2 M_{xx} - \frac{1}{2}b^2(1-p)^2\rho^2 M_{xx} - \frac{b(1-p)\rho(\mu - r)}{\sigma}M_x = 0 \qquad (11)$$

with $M(T, x) = U(x)$. Then, differentiating with respect to $p(t)$ in (11) and we have

$$\begin{aligned}
-\eta a M_x - b^2(1-p)M_{xx} + b^2(1-p)\rho^2 M_{xx} + \frac{b\rho(\mu - r)}{\sigma}M_x = 0 \\
\Leftrightarrow -\eta a\, M_x - b^2(1-p)M_{xx}(1+\rho^2) + \frac{b\rho(\mu - r)}{\sigma}M_x = 0 \\
\Leftrightarrow b^2 p(1+\rho^2)M_{xx} = \eta a M_x + b^2(1+\rho^2)M_{xx} - \frac{b\rho(\mu - r)}{\sigma}M_x \\
\Leftrightarrow p^* = 1 + \frac{\eta a}{b^2(1+\rho^2)}\frac{M_x}{M_{xx}} - \frac{\rho(\mu - r)}{\sigma b(1+\rho^2)}\frac{M_x}{M_{xx}}.
\end{aligned}$$

In the same way with Theorem 1, plugging these derivatives M_t , M_x and M_{xx} into the HJB equation (10), we have

$$\begin{aligned}
x^\alpha f'(t) + \Bigg(rx - \frac{(\mu - r)^2 x}{\sigma^2\,(\alpha - 1)} - \frac{b(1-p)\rho(\mu - r)}{\sigma} + a\theta - \eta a + \frac{\eta^2 a^2 x}{b^2(1+\rho^2)(\alpha - 1)} \\
- \frac{\eta\,\rho(\mu - r)x\,a}{\sigma b\,(1+\rho^2)(\alpha - 1)}\Bigg)\alpha x^{\alpha - 1} f(t) \\
+ \frac{1}{2}\Bigg\{\frac{(\mu - r)^2 x^2}{\sigma^2(\alpha - 1)^2} + b^2(1-p)^2\rho^2 + \frac{2(\mu - r)xb(1-p)\rho}{\sigma(\alpha - 1)} + \frac{\eta^2 a^2 x^2}{b^2(1+\rho^2)^2(\alpha - 1)^2} \\
+ \frac{\rho^2(\mu - r)^2 x^2}{\sigma^2(1+\rho^2)^2(\alpha - 1)^2} - \frac{2\eta a x^2\rho(\mu - r)}{(1+\rho^2)^2(\alpha - 1)^2\sigma b} \\
+ 2\left(\frac{(\mu - r)^2 x}{\sigma^2\,(\alpha - 1)} - \frac{b(1-p)\rho(\mu - r)}{\sigma}\right)\sigma\, b\, \Bigg(-\frac{\eta a\, x}{b^2(1+\rho^2)(\alpha - 1)} \\
+ \frac{\rho(\mu - r)x}{\sigma\, b\,(1+\rho^2)(\alpha - 1)}\Bigg)\rho\Bigg\}\alpha(\alpha - 1)x^{\alpha - 2} f(t) = 0
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow f'(t) + \Bigg\{\Bigg[rx + \frac{\left(-2\left(\rho^2 + \frac{1}{2}\right)(\mu - r)b + a\rho\eta\sigma\right)x(\mu - r)}{b(\alpha - 1)(1+\rho^2)\sigma^2} \\
+ \left(\frac{-\left((\alpha - 1)b^2(1+\rho^2)\sigma + \eta x\, a\,\sigma - \rho(\mu - r)xb\right)\eta}{(\alpha - 1)b^2(1+\rho^2)\sigma} + \theta\right)a\Bigg]\frac{\alpha}{x} \\
+ \frac{1}{2}\Bigg[\frac{\left(-2\left(\rho^2 + \frac{1}{2}\right)(\mu - r)b + a\rho\eta\sigma\right)^2 x^2}{b^2(\alpha - 1)^2(1+\rho^2)\sigma^2} \\
+ b^2\left(1 - \frac{(\alpha - 1)b^2(1+\rho^2)\sigma + \eta x\, a\,\sigma - \rho(\mu - r)x\, b}{(\alpha - 1)b^2(1+\rho^2)\sigma}\right)^2 \\
+ \frac{2\left(-2\left(\rho^2 + \frac{1}{2}\right)(\mu - r)b + a\rho\eta\sigma\right)x\left(1 - \frac{(\alpha - 1)b^2(1+\rho^2)\sigma + \eta x a\sigma - \rho(\mu - r)xb}{(\alpha - 1)b^2(1+\rho^2)\sigma}\right)\rho}{(\alpha - 1)(1+\rho^2)\sigma}\Bigg]\frac{\alpha(\alpha - 1)}{x^2}\Bigg\} f(t).
\end{aligned}$$

The solution of the above ordinary differential equation is

$$f(t) = K_1 \exp\left(\left(\left[rx + \frac{\left(-2\left(\rho^2+\frac{1}{2}\right)(\mu-r)b + a\rho\eta\sigma\right)x(\mu-r)}{b(\alpha-1)(1+\rho^2)\sigma^2} + \left(\frac{-\left((\alpha-1)b^2(1+\rho^2)\sigma + \eta x\, a\, \sigma - \rho(\mu-r)xb\right)\eta}{(\alpha-1)b^2(1+\rho^2)\sigma} + \theta\right)a\right]\frac{\alpha}{x} + \frac{1}{2}\left[\frac{\left(-2\left(\rho^2+\frac{1}{2}\right)(\mu-r)b + a\rho\eta\sigma\right)^2 x^2}{b^2(\alpha-1)^2(1+\rho^2)\sigma^2} + b^2\left(1-\frac{(\alpha-1)b^2(1+\rho^2)\sigma + \eta x\, a\, \sigma - \rho(\mu-r)x\, b}{(\alpha-1)b^2(1+\rho^2)\sigma}\right)^2 + \frac{2\left(-2\left(\rho^2+\frac{1}{2}\right)(\mu-r)b + a\rho\eta\sigma\right)x\left(1-\frac{(\alpha-1)b^2(1+\rho^2)\sigma+\eta x a\sigma-\rho(\mu-r)xb}{(\alpha-1)b^2(1+\rho^2)\sigma}\right)\rho}{(\alpha-1)(1+\rho^2)\sigma}\right]\frac{\alpha(\alpha-1)}{x^2}\right)t\right).$$

for $K_1 \geq 0$.

So, we have the value function is

$$M(t,x) = x^\alpha K_1 \exp\left(\left(\left[rx + \frac{\left(-2\left(\rho^2+\frac{1}{2}\right)(\mu-r)b + a\rho\eta\sigma\right)x(\mu-r)}{b(\alpha-1)(1+\rho^2)\sigma^2} + \left(\frac{-\left((\alpha-1)b^2(1+\rho^2)\sigma + \eta x\, a\, \sigma - \rho(\mu-r)xb\right)\eta}{(\alpha-1)b^2(1+\rho^2)\sigma} + \theta\right)a\right]\frac{\alpha}{x} + \frac{1}{2}\left[\frac{\left(-2\left(\rho^2+\frac{1}{2}\right)(\mu-r)b + a\rho\eta\sigma\right)^2 x^2}{b^2(\alpha-1)^2(1+\rho^2)\sigma^2} + b^2\left(1-\frac{(\alpha-1)b^2(1+\rho^2)\sigma + \eta x\, a\, \sigma - \rho(\mu-r)x\, b}{(\alpha-1)b^2(1+\rho^2)\sigma}\right)^2 + \frac{2\left(-2\left(\rho^2+\frac{1}{2}\right)(\mu-r)b + a\rho\eta\sigma\right)x\left(1-\frac{(\alpha-1)b^2(1+\rho^2)\sigma+\eta x a\sigma-\rho(\mu-r)xb}{(\alpha-1)b^2(1+\rho^2)\sigma}\right)\rho}{(\alpha-1)(1+\rho^2)\sigma}\right]\frac{\alpha(\alpha-1)}{x^2}\right)t\right)$$

and the optimal strategy $\beta^* = \left((\pi^*, p^*)\right)$ as follows:

$$\pi^* = \frac{\left(-2\left(\rho^2+\frac{1}{2}\right)(\mu-r)b + a\rho\eta\sigma\right)x}{b(\alpha-1)(1+\rho^2)\sigma^2}$$

and

$$p^* = \frac{(\alpha-1)b^2(1+\rho^2)\sigma + \eta x\, a\, \sigma - \rho(\mu-r)xb}{b^2(\alpha-1)(1+\rho^2)\sigma}.$$

4. Conclusions

In this paper, we focus on the optimal reinsurance and investment problem for an insurer under a fractional power utility function. The basic claim process is assumed to follow a Brownian motion with drift, and the insurer could purchase proportional reinsurance. Otherwise, the insurer allowed to invest in a risk-free asset and a risky asset that follows the Black Scholes model. Firstly, we consider the optimization problem of maximizing expected fractional power utility with standard Brownian motion from the claim process, and the price of the risky asset is independent. Secondly, we consider standard Brownian motion from the claim process, and the price of the risky asset is independent is dependent. Finally, by solving the corresponding HJB equation, we derive the optimal reinsurance and investment strategy explicitly.

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