The Equilibrium Solution of Word-of-Mouth Marketing Strategy

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Abstract

There are several marketing strategies known in literature, such as advertising, display in a shop, distribution channels, promotion, etc. Current form of marketing strategy include the use of information technology, such as website and social media. Related to the later, other specific methods, such as word of mouth (WOM) and celebrity endorsment are becoming popular. The WOM communications is considered to be have advantages such as significantly lower cost and much faster propagation of the messages. In this paper we will present a mathematical model for word of mouth marketing strategy by considering proportional recruitment. We divide a population under consideration into four subpopulations: susceptible – those who are the target market or potential buyers (S), infected – those who are already active as buyers (I), positive – former buyers which have positive comments on the item they purchased (P) and negative – former buyers which have negative comments on the item they purchased (N). We assume that the rate of new individuals who enter the target market is proportional to the number of each subpopulations, i. e. S, I, P, and N. They have either a positive or negative contribution to the number of new entry to the susceptible class or the potential buyer. We analyzed the model and found the condition for the equilibrium coexistence solution to exist in the long-term.

Keywords (12 font)
Marketing Strategy, Word-of-Mouth, Mathematical Model, Dynamical System, Equilibrium Coexistence Solution.

1. Introduction

Recently Li et al. (2018) proposes a mathematical model and analysis the model of the effect word-of-mouth (WOM) on the dynamics of buying behavior or marketing strategy. Sometimes WOM is called WOMM (word-of-mouth marketing) strategy. In essence, the strategy uses a "free adverstisement" given by satisfactory former and current buyers who express their satisfactory to other persons. This is looks looks like a "friend get friend" schemes. This strategy in general is difficult to manage. However Lang and Hide (2013) argue that there are three actions that may help in managing the WOM: 1) Developing a strong WOM foundation, such as sufficient levels of satisfaction, trust and commitment, 2) Undertaking indirect WOM management, such as customer membership clubs, 3) Undertaking direct WOM management, such as paid WOM 'agents', "friend get friend" schemes.

Li et al. (2018). Developed a mathematical model by considering several human subpopulations related to a product under consideration. They called the model as SIPNS model, that characterizes the WOM marketing processes with both positive and negative comments which is established by former buyers. The main variables in the model are four denote the number of susceptible (*S*), infected (*I*), positive (*P*) and negative (*N*) individuals at time *t.* The model governed by a system of differential equations, describing the growth of those subpopulations. Let the susceptible subpopulation consists of potential buyers (with the size at time *t* is denoted by $S(t)$, $t\geq 0$), the infected subpopulation consits of active buyers (with the size at time *t* is denoted by $I(t)$, $t\geq 0$), the positive subpopulation nconsists of former buyer who have positive comment regarding the product or other factors related to the buying process (with the size at time *t* is denoted by $P(t)$, $t\geq 0$), and the negative subpopulation consists of former buyers with negative comments (with the size at time *t* is denoted by $N(t)$, $t\geq 0$). Using these variables, the model is given by

$$\frac{dS(t)}{dt} = \mu - \beta_P P(t)S(t) - \beta_N N(t)S(t) + \gamma_P P(t) + \gamma_I I(t),$$

$$\frac{dI(t)}{dt} = \beta_P P(t)S(t) - \alpha_P I(t) - \alpha_N I(t) - \gamma_I I(t) - \delta_I I(t),$$

$$\frac{dP(t)}{dt} = \alpha_P I(t) - \gamma_P P(t) - \delta_P P(t),$$

$$\frac{dN(t)}{dt} = \alpha_N I(t) - \delta_N N(t),$$

with the description of parameters are given in Table 1.

Table 1. The Description of Parameters (Li et al. 2018)

Parameter	Definition
μ	the average rate of new individuals enter the target market and become susceptible
$\boldsymbol{\beta_P}$	the *P*-infection force or the force or encouragement by the positive comments for a susceptible individual to purchase an item
$\boldsymbol{\beta_N}$	the *N*-infection force or discouragement by the negative comments for a susceptible individual to exit from the market
$\boldsymbol{\gamma_P}$	the *P*-viscosity rate: due to the shopping desire, a positive individual tends to purchase one more item and hence becomes susceptible
$\boldsymbol{\gamma_I}$	the *I*-viscosity rate: due to the shopping desire, an infected individual tends to purchase one more item and hence becomes susceptible
$\boldsymbol{\alpha_P}$	the *P*-comment rate: due to the desire to express the feeling for the recently purchased item, at any time an infected individual makes a positive comment on the item and hence becomes positive
$\boldsymbol{\alpha_N}$	the *N*-comment rate: due to the desire to express the feeling for the recently purchased item, at any time an infected individual makes a negative comment on the item and hence becomes negative
$\boldsymbol{\delta_I}$	*I*-exit rate: due to the loss of interest in shopping, at any time an infected individual exits from the target market
$\boldsymbol{\delta_P}$	*P*-exit rate: due to the loss of interest in shopping, at any time a P individual exits from the target market
$\boldsymbol{\delta_N}$	*N*-exit rate: due to the loss of interest in shopping, at any time an N individual exits from the target market

The model resembles the spead of disease in epidemiology of SEIR type - susceptible, exposed, infected, and recovered (Nainggolan, 2013), giving up smoking dynamic (Chot et al., 2011), and the spread of malware in computers (Ndii et al., 2019; Lanz et al. 2019). The author of the model (Li et al., 2018) obtains the equilibrium solution in the form (S^*, I^*, P^*, N^*) and establish a theorem showing that S^* is irrelevant to μ, but I^*, P^* and $N^*$are strictly increasing with μ. The effects of other parameters are also investigated and giving emphasize in the impact of the two comment rates into this non-trivial equilibrium (S^*, I^*, P^*, N^*) and the expected overall profit of a WOM marketing campaign.

We note that in discussing the expected overall profit they calculated from the potential buyers not from the actual buyers. They also develop the model by assuming a constant rate of new individuals who enter the target market and become susceptible. In the next section we develop a model that assumes the rate of new individuals who enter the target market is proportional to the number of each subpopulations, i. e. *S*, *I*, *P*, and *N*. They have either a positive or negative contribution to the number of new entry to the susceptible class or the potential buyer.

2. Mathematical Model

As in Li et al. (2018) let the susceptible subpopulation consists of potential buyers (with the size at time *t* is denoted by *S*(*t*), *t*≥0), the infected subpopulation consits of active buyers (with the size at time *t* is denoted by *I*(*t*), *t*≥0), the positive subpopulation consists of former buyer who have positive comment regarding the product or other factors related to the buying process (with the size at time *t* is denoted by *P*(*t*), *t*≥0), and the negative subpopulation consists of former buyers with negative comments (with the size at time *t* is denoted by *N*(*t*), *t*≥0). Further we assume that there is no demographic factors in the model, since we will give attention on the effect of population sizes of *S*, *I*, *P*, and *N* to the dynamics of the model. By using these assumptions, the model is given by

$$\frac{dS(t)}{dt} = \mu(S + I + P - N) - \beta_P P(t)S(t) - \beta_N N(t)S(t) + \gamma_P P(t),$$

$$\frac{dI(t)}{dt} = \beta_P P(t)S(t) - \alpha_P I(t) - \alpha_N I(t),$$

$$\frac{dP(t)}{dt} = \alpha_P I(t) - \gamma_P P(t),$$

$$\frac{dN(t)}{dt} = \alpha_N I(t) - \delta_N N(t),$$

with the description of parameters are the same as Table 1.
The equilibrium of the system is given by (S^*, I^*, P^*, N^*) where:

$$S^* = \frac{\gamma_P(\alpha_P + \alpha_N)}{\beta_P \alpha_P},$$

$$I^* = \frac{\mu(\gamma_P)^2(\alpha_P + \alpha_N)}{-\mu\gamma_P\beta_P\alpha_P - \mu\beta_P(\alpha_P)^2 + \mu\beta_P\alpha_P\alpha_N + \gamma_P\beta_P\alpha_P\alpha_N + \gamma_P\beta_N\alpha_P\alpha_N + \gamma_P\beta_N(\alpha_N)^2},$$

$$P^* = \frac{\alpha_P\mu\gamma_P(\alpha_P + \alpha_N)}{-\mu\gamma_P\beta_P\alpha_P - \mu\beta_P(\alpha_P)^2 + \mu\beta_P\alpha_P\alpha_N + \gamma_P\beta_P\alpha_P\alpha_N + \gamma_P\beta_N\alpha_P\alpha_N + \gamma_P\beta_N(\alpha_N)^2},$$

$$N^* = \frac{\alpha_N\mu\gamma_P(\alpha_P + \alpha_N)}{-\mu\gamma_P\beta_P\alpha_P - \mu\beta_P(\alpha_P)^2 + \mu\beta_P\alpha_P\alpha_N + \gamma_P\beta_P\alpha_P\alpha_N + \gamma_P\beta_N\alpha_P\alpha_N + \gamma_P\beta_N(\alpha_N)^2},$$

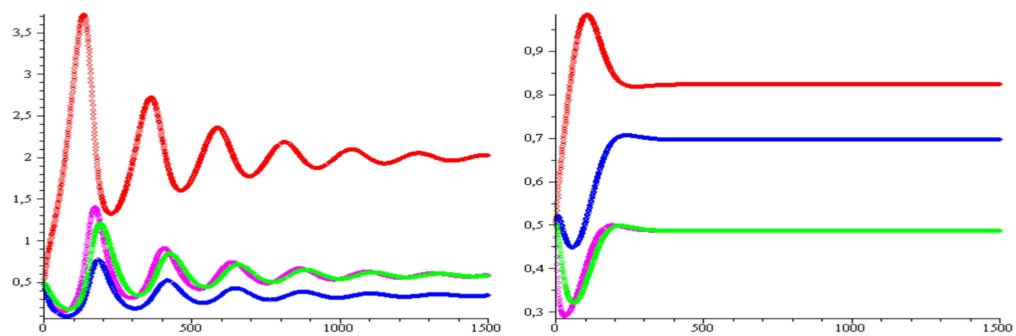
which among them can be simplified to:
$I^* = \gamma_P(R_0 - 1)$, $P^* = \alpha_P(R_0 - 1)$, and $N^* = \alpha_N(R_0 - 1)$, where $R_0 = \frac{\mu\gamma_P(\alpha_P+\alpha_N)+(\mu\alpha_P\beta_P(\alpha_N-\gamma_P-\alpha_P)+\alpha_N\gamma_P(\alpha_P(\beta_P+\beta_N)+\alpha_N\beta_N))}{\mu\alpha_P\beta_P(\alpha_N-\gamma_P-\alpha_P)+\alpha_N\gamma_P(\alpha_P(\beta_P+\beta_N)+\alpha_N\beta_N)}$.

Clearly all the equilibrium subpopulations will be positive whenever $R_0 > 1$. This property resembles the fundamental theorem in epidemiology. We call it as the basic reproduction number. The number is very important in determining the existence of positive equilibrium as shown in the following theorem

Theorem 1:
In the long-term there exists a co-existence equilibrium (S^*, I^*, P^*, N^*) where all the subpopulations does not extinct provided $R_0 > 1$.

Proof:
It is easy to prove since in this case $I^* = \gamma_P(R_0 - 1) > 0$, $P^* = \alpha_P(R_0 - 1) > 0$, $N^* = \alpha_N(R_0 - 1) > 0$, and S^* is already positive from the first instance.

As an illustration of the theorem above we give a numerical example in Figure 1. The computation uses the Runge-Kutta method of order four. The example uses the following values of parameters: μ=0.2, β_P=0.85, β_N=0.15, γ_P=0.85, α_P=0.5, α_N=0.5, δ_N=0.05 (data set 1) and μ=0.2, β_P=0.85, β_N=0.15, γ_P=0.35, α_P=0.5, α_N=0.5, δ_N=0.15 (data set 2) with the initial values $(S^0 = 0.5, I^0 = 0.5, P^0 = 0.5, N^0 = 0.5)$. Figure 1 shows the solution of the model in which all subpopulations approach the stable equilibrium (S^*, I^*, P^*, N^*) in the long-term as indicated by Theorem 1. The left figure (a) uses the data set 1 resulting in $(S^*, I^*, P^*, N^*) = (2.0, 0.59, 0.34, 0.58)$ and the right figure (b) uses the data set 2 resulting in $(S^*, I^*, P^*, N^*) = (0.82, 0.49, 0.70, 0.49)$. The number are in thousands. Both data sets resulting the basic reproduction number greater than one, i.e. $R_0 = 1.516129032$ and $R_0 = 1.377104377$ as expected.



(a) (b)

Figure 1. The solution of the model. Horizontal axes is time (t) and vertical axes is the size of subpopulation. The figure show the solution for *S* (red), *I* (magenta), *P* (blue), and *N* (green) for different data sets.

3. Conclusion

In this paper we have developed a mathematical model for word of mouth marketing strategy by considering proportional recruitment. The results indicate that there is a coexistence equilibrium solution and we found the condition for the equilibrium to exist in the long-term. This condition is represented by the threshold number, called the basic reproduction number, which is very important in determining the long-term number of active and potential buyers.

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