Adomian Decomposition Method and The New Integral Transform

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Abstract

The Adomian decomposition method is an iterative method that can be used to solve integral, differential, and integrodifferential equations. The differential equations that can be solved by this method can be of integer or fractional order, ordinary or partial, with initial or boundary value problems, with variable or constant coefficients, linear or nonlinear, homogeneous or nonhomogeneous. This method divides the equation into two forms, namely linear and nonlinear, so that it can solve equations without linearization, discretization, perturbation, or other restrictive assumptions. The basic concept of this method assumes that the solution can be decomposed into an infinite series. In particular, this method decomposes the nonlinear form (if any) of the equation with the Adomian polynomial series. This decomposition method can be combined with various integral transform, such as Laplace, Sumudu, Elzaki, and Mohand. The main idea of this technique assumes that the solution can be decomposed into an infinite series, then applies the integral transform to the differential equation. The main advantage of this technique is that the solution can be expressed as an infinite series that converges rapidly to the exact solution. This paper aims to combine the Adomian decomposition method with the new integral transform introduced by Kashuri and Fundo (2013). A scheme for solving fractional ordinary differential equations using the combined method is presented in this paper.

Keywords
Adomian decomposition method, new integral transform, fractional ordinary differential equation.

1. Introduction
The Adomian decomposition method was first introduced by George Adomian to solve a system of stochastic equations (Adomian, 1980). This method can be an effective procedure to obtain analytical solutions without linearization or weak nonlinear assumptions, perturbation, discretization, or restrictive assumptions in stochastic cases (Adomian, 1988). The Adomian decomposition method can be used to solve integral, differential, and integral-differential equations. Differential equations that can be solved by this decomposition method can be of integer or fractional order, ordinary or partial, with initial or boundary value problems, with variable or constant coefficients, linear or nonlinear, homogeneous or nonhomogeneous. This decomposition method is also able to solve algebraic equations, delay differential equations, and equation systems (Duan et al., 2012; Al-awawdah, 2016; Sumiati et al., 2019).

The basic concept of this method assumes that the solution is decomposed into an infinite series, the nonlinear form is decomposed into Adomian polynomials, and an iterative algorithm is constructed to recursively determine the solution. The Adomian decomposition method is a powerful and useful technique for solving heat equations (Biazar & Amirtaimoori, 2005), and waves (Luo et al., 2006). Combined with the Caputo derivative, the Adomian decomposition method can solve diffusion differential equations, waves (Jafari & Daftardar-Gejji, 2006), and Burgers (Gepreel, 2012) with fractional order.

The numerical scheme for the Laplace transform based on the Adomian decomposition method can be used to obtain an approximate solution to nonlinear differential equations. The main idea of this technique assumes that the solution can be decomposed into an infinite series, then applies the Laplace transform to the differential equation. The main advantage of this technique is that the solution can be expressed as an infinite series that converges rapidly to the exact solution (Khuri, 2001). The Adomian-Laplace decomposition method is used for solving the Volterra integrodifferential equation (Wazwaz, 2010) and non-homogeneous heat equations that arise in fractal heat flow (Jassim, 2015).

Other integral transforms that can be combined with the Adomian decomposition method are the Sumudu, Elzaki, and Mohand transforms. These transforms have similar properties to the Laplace transform but is simpler. The Adomian-Sumudu decomposition method can be used to solve ordinary (Khan et al., 2008) and partial (Kumar et al., 2012b) differential equations with natural number orders. This method can also effectively provide a solution to the fractional Riccati equation (Mahdy & Marai, 2018), where the fractional derivative used is Caputo.

Elzaki and Alkhateeb (2015) applied the Adomian decomposition method to calculate the Elzaki transform of several functions. This method can easily be generalized to calculate the Elzaki transform for many other functions and also the results obtained can be used to solve nonlinear equations. Mahgoub and Sedeeg (2016) apply the combined form of the Elzaki transform method with the Adomian decomposition method to obtain a solution to the Newell-Whitehead Seal equation. Mohamed and Elzaki (2020) conducted a study to obtain an approximate resolution of fractional partial differential equations through the help of a new integral transform called the Elzaki decomposition method. The Adomian-Mohand decomposition method can be used to solve systems of order-fractional partial differential equations (Shah et al., 2019), third-order Kortewege-De Vries equations with fractional order (Shah et al., 2020), and time-fractional telegraph equations (Ali et al., 2020).

A new integral transform is derived from the classical Fourier integral. This new integral transform was introduced by Kashuri and Fundo (2013) to facilitate the process of solving ordinary and partial differential equations in the time domain. Typically, the Fourier, Laplace, Sumudu, and Elzaki transforms are easy mathematical tools to solve differential equations. This new integral transform also has some of the basic properties used to solve differential equations such as other integral transforms.

Therefore, based on the background of the problems and previous studies that have been presented, this paper aims to combine the Adomian decomposition method with the new integral transform introduced by Kashuri and Fundo. A schematic for solving ordinary differential equations with fractional order using the combined method is presented in this paper.

2. Literature Review

This section presents the basic theories and concepts related to fractional calculus and Kashuri-Fundo transform.

Definition 1. (Kashuri & Fundo, 2013) Given a set of functions

$$F = \left\{ f(t): \exists M, k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{k_j^2}}, t \in (-1)^j \times [0, \infty) \right\},$$

Kashuri-Fundo transform is defined as

$$\mathcal{K}[f(t)] = A(v) = \frac{1}{v} \int_0^\infty e^{-\frac{t}{v^2}} f(t) dt, t \geq 0, -k_1 < v < k_2.$$

Next, the inverse of Kashuri-Fando transform is denoted by $\mathcal{K}^{-1}[A(v)] = f(t)$, $t \geq 0$. For α is a fractional number, valid

$$\mathcal{K}[t^{\alpha}] = \Gamma(\alpha + 1)v^{2\alpha+1}. \tag{1}$$

Definition 2. (Podlubny, 1999; Mathai & Haubold, 2017) The Caputo fractional derivative of the function f with t in the order α, where $\alpha > 0$, is defined as

$$ {}_{a}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-u)^{n-\alpha-1} f^{(n)}(u)du\,, n-1 < \alpha \leq n.$$

Definition 3. (Kashuri & Fundo, 2013) The Kashuri transform of Caputo fractional derivative for $a = 0$ is defined as

$$\mathcal{K}[{}^{C}D_t^{\alpha} f(t)] = \frac{A(v)}{v^{2\alpha}} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{v^{2(\alpha-k)-1}}, n-1 < \alpha \leq n.$$

3. Kashuri-Fundo Decomposition Method

This section presents the solution of fractional ordinary differential equations using the Kashuri-Fundo decomposition method.

Given the fractional ordinary differential equation as follows

$$D_t^{\alpha} y(t) = g(t) + Ny(t) + Ry(t), \tag{2}$$

and initial condition $y(0) = c$, where $D_t^{\alpha} \equiv {}^{C}D_t^{\alpha}$ is a Caputo fractional derivative operator with $0 < \alpha \leq 1$, N is a nonlinear operator, R is a linear operator, g is a function that shows the homogeneity of the differential equation, and y is a function of t to be determined. Using the Kashuri-Fundo transform in equation (2), thus based on Definition 3, is obtained

$$y(v) = vy(0) + v^{2\alpha}\mathcal{K}[g(t)] + v^{2\alpha}\mathcal{K}[Ny(t)] + v^{2\alpha}\mathcal{K}[Ry(t)]. \tag{3}$$

Next, using the inverse of Kashuri-Fundo transform in equation (3), is obtained

$$y(t) = y(0) + \mathcal{K}^{-1}\left[v^{2\alpha}\mathcal{K}[g(t)]\right] + \mathcal{K}^{-1}\left[v^{2\alpha}\mathcal{K}[Ny(t)]\right] + \mathcal{K}^{-1}\left[v^{2\alpha}\mathcal{K}[Ry(t)]\right]. \tag{4}$$

The Adomian decomposition method assumes that the y function can be broken down or decomposed into an infinite series (Adomian, 1988; Al Awadah, 2016)

$$y(t) = \sum_{n=0}^{\infty} y_n(t) = y_0 + y_1 + y_2 + \cdots, \tag{5}$$

where y_n can be specified recursively. This method also assumes the nonlinear operator Ny can be decomposed into an infinite polynomial series

$$Ny = \sum_{n=0}^{\infty} A_n, \tag{6}$$

where $A_n = A_n(y_0, y_1, y_2, \dots, y_n)$ is a defined Adomian polynomial,

$$A_n(y_0, y_1, y_2, \dots, y_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{k=0}^{n}\lambda^k y_k\right)\right]_{\lambda=0}, n = 0,1,2, \dots$$

where λ is a parameter. The Adomian polynomial A_n can be described as follows

$$\begin{aligned}
A_0 &= \frac{1}{0!}\frac{d^0}{d\lambda^0}\left[N\left(\sum_{k=0}^{0}\lambda^k y_k\right)\right]_{\lambda=0} = N(y_0),\\
A_1 &= \frac{1}{1!}\frac{d^1}{d\lambda^1}\left[N\left(\sum_{k=0}^{1}\lambda^k y_k\right)\right]_{\lambda=0} = y_1 N'(y_0),\\
A_2 &= \frac{1}{2!}\frac{d^2}{d\lambda^2}\left[N\left(\sum_{k=0}^{2}\lambda^k y_k\right)\right]_{\lambda=0} = y_2 N'(y_0) + \frac{y_1^2}{2!}N''(y_0),
\end{aligned}$$

Substitute equations (5) and (6) to equation (4), obtained

$$\sum_{n=0}^{\infty} y_n(t) = y(0) + \mathcal{K}^{-1}\left[v^{2\alpha}\mathcal{K}[g(t)]\right] + \mathcal{K}^{-1}\left[v^{2\alpha}\mathcal{K}\left[\sum_{n=0}^{\infty} A_n\right]\right] + \mathcal{K}^{-1}\left[v^{2\alpha}\mathcal{K}\left[R\sum_{n=0}^{\infty} y_n(t)\right]\right]. \tag{7}$$

If both sides of equation (7) are described, then successively is obtained

$$\begin{aligned}
y_0 &= y(0) + \mathcal{K}^{-1}\left[v^{2\alpha}\mathcal{K}[g(t)]\right],\\
y_1 &= \mathcal{K}^{-1}\left[v^{2\alpha}\mathcal{K}[A_0]\right] + \mathcal{K}^{-1}\left[v^{2\alpha}\mathcal{K}[Ry_0]\right],\\
y_2 &= \mathcal{K}^{-1}\left[v^{2\alpha}\mathcal{K}[A_1]\right] + \mathcal{K}^{-1}\left[v^{2\alpha}\mathcal{K}[Ry_1]\right],\\
y_3 &= \mathcal{K}^{-1}\left[v^{2\alpha}\mathcal{K}[A_2]\right] + \mathcal{K}^{-1}\left[v^{2\alpha}\mathcal{K}[Ry_2]\right],\\
&\vdots
\end{aligned}$$

thus generally obtained the recursive relation of the fractional ordinary differential equation solution (2) using the Kashuri-Fundo decomposition method as follows

$$\begin{aligned}
y_0 &= y(0) + \mathcal{K}^{-1}\left[v^{2\alpha}\mathcal{K}[g(t)]\right],\\
y_{n+1} &= \mathcal{K}^{-1}\left[v^{2\alpha}\mathcal{K}[A_n]\right] + \mathcal{K}^{-1}\left[v^{2\alpha}\mathcal{K}[Ry_n]\right], n = 0,1,2, \dots.
\end{aligned} \tag{8}$$

Therefore, the approximate solution of the fractional ordinary differential equation (2) using the Kashuri-Fando decomposition method is

$$y \approx \sum_{n=0}^{k} y_n, \text{where } \lim_{k\to\infty}\sum_{n=0}^{k} y_n = y.$$

4. Conclusion

The Kashuri-Fundo decomposition method is a combination of the Adomian decomposition method and the Kashuri-Fundo integral transform. This paper presents a scheme for solving fractional ordinary differential equations using the Kashuri-Fundo decomposition method.

Acknowledgments

Acknowledgments are conveyed to the Director of General of Higher Education of the Republic of Indonesia, and Chancellor, Director of the Directorate of Research, Community Engagement, and Innovation, and the Dean of the Faculty of Mathematics and Natural Sciences, Universitas Padjadjaran, who have provided the Basic Research Scheme Grant with Contract Number: **1827/UN6.3.1/LT/2020.**

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