Stochastic Analysis of a Three Unit Series Parallel System

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Abstract

A three unit series parallel system is considered. How the introduction of realistic assumptions in the model complicates the behaviour of the system leading to the identification of its underlying behaviour as a Renewal process, Alternating renewal Process, Markov Process, Markov Renewal Process, Semi Markov Processes and a Regenerative Stochastic Processes is explained. Some of the system measures are obtained in each of these cases.

1 Basic System

We consider a system consisting of 3 units in which 2 units Unit 2 and Unit 3 are connected in parallel and this parallel system is connected in series with Unit 1 as shown in Figure 1.

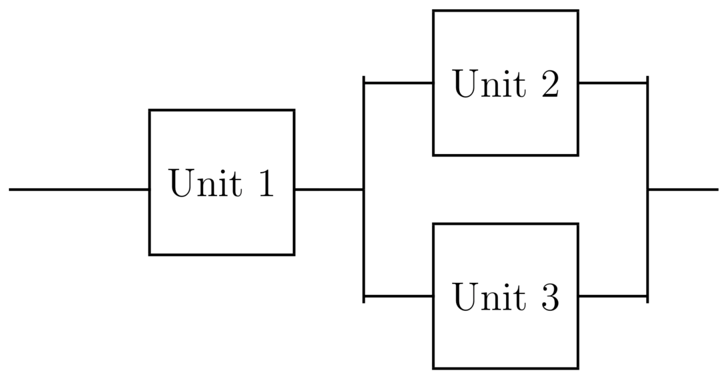

Figure 1: Birolini (1971)

2 Model 1 - RENEWAL PROCESS

First we consider the system with the following assumptions

2.1 Assumptions

1. All the three units are statistically independent and identical and have the same failure rate (FR) λ. When a unit fails it is instantaneously replaced with a similar new unit.

2.2 Analysis

We are interested in finding the number of replacements in the time interval $(0, t]$. We note that the underlying Stochastic Process is a Renewal Process with the pdf of the interval between two successive renewals (replacements) having an exponential distribution with mean $\frac{1}{3\lambda}$.The renewal function is given by $3\lambda t$.

2.3 Assumptions

1. The life time (LT)of Unit 1 is a random variable X with an arbitrary distribution having pdf $f(.)$ .Unit 2 and Unit 3 are statistically independent and identical and have the same failure rate λ
2. When Unit 1 fails it is instantaneously taken up for repair and the repair time (RT) is a random variable Y with an arbitrary distribution having pdf $g(.)$. When Unit2 and Unit 3 fail they are instantaneously replaced with a similar new unit.

2.4 Analysis

We are interested in the availability of the system. Since Unit 2 and Unit 3 are instantaneously replaced they can be considered as available at all times and the availability of the system is determined by the availability of Unit 1 alone . Let E_0 denote the event that Unit 1 is just online and E_1 the event that repair just commences for Unit 1. Then we note that E_0 and E_1 occur alternately and constitute an renewal process.

Let $Z(t)$ denote the state of the system (0 operable and 1 failed) at time t. Define $A_i(t) =$ Pr $\{ Z(t) = 0|E_i$ at $t = 0\}, i = 0, 1$ Then using probabilistic arguments we derive the following equation:

$$A_0(t) = \bar{F}(t) + \sum_{n=1}^{\infty} \{f(t)©g(t)\}^{(n)}©\bar{F}(t)$$

The steady state availability of the system defined as

$$A_\infty = \lim_{t\to\infty} A_0(t) = \lim_{t\to\infty} A_1(t)$$

is given by

$$A_\infty = \frac{E(X)}{E(X) + E(Y)}.$$

Special cases: Let $f(t) = \lambda e^{-\lambda t}$ and $g(t) = \mu e^{-\mu t}$
Then

$$A_0^*(s) = \frac{s + \mu}{s(s + \lambda + \mu)}$$

$$A_0(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}$$

and

$$A_\infty = \frac{\mu}{\lambda + \mu} = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + \frac{1}{\mu}} = \frac{E(X)}{E(X + Y)}$$

3 MODEL 2 - ALTERNATING RENEWAL PROCESS

3.1 Assumptions

1. The life time (LT)of Unit 1 is a random variable X with an arbitrary distribution having pdf $f(.)$ .Unit 2 and Unit 3 are statistically independent and identical and have the same failure rate λ
2. When Unit 1 fails it is instantaneously taken up for repair and the repair time (RT) is a random variable Y with an arbitrary distribution having pdf $g(.)$. When Unit2 and Unit 3 fail they are instantaneously replaced with a similar new unit.

3.2 Analysis

We are interested in the availability of the system. Since Unit 2 and Unit 3 are instantaneously replaced they can be considered as available at all times and the availability of the system is determined by the availability of Unit 1 alone . Let E_0 denote the event that Unit 1 is just online and E_1 the event that repair just commences for Unit 1. Then we note that E_0 and E_1 occur alternately and constitute an renewal process. Let $Z(t)$ denote the state of the system (0 operable and 1 failed) at time t. Define $A_i(t) =$ Pr $\{ Z(t) = 0|E_i$ at $t = 0\}, i = 0, 1$ Then using probabilistic arguments we derive the following equation:

$$A_0(t) = \bar{F}(t) + \sum_{n=1}^{\infty} \{f(t)©g(t)\}^{(n)}©\bar{F}(t)$$

The steady state availability of the system defined as

$$A_\infty = \lim_{t\to\infty} A_0(t) = \lim_{t\to\infty} A_1(t)$$

is given by

$$A_\infty = \frac{E(X)}{E(X)+E(Y)}.$$

Special cases: Let $f(t) = \lambda e^{-\lambda t}$ and $g(t) = \mu e^{-\mu t}$
Then

$$A_0^*(s) = \frac{s+\mu}{s(s+\lambda+\mu)}$$

$$A_0(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

and

$$A_\infty = \frac{\mu}{\lambda+\mu} = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda}+\frac{1}{\mu}} = \frac{E(X)}{E(X+Y)}$$

4 MODEL 3 - MARKOV PROCESS

4.1 Assumptions

1. Unit 2 and Unit 3 are statistically identical
2. The failure rate of Unit 1 is λ_1. Unit 2 and Unit 3 have the same failure rate λ_2
3. There is a single repair facility and the repair rate of Unit 1 is μ_1 and that of Unit 2 and Unit 3 is μ_2. Preemptive priority is followed for the repair of Unit 1.
4. Operable units cannot fail when the system is in the down state.

4.2 Notation

The state of the system $Z(t)$ at any time t is is represented by the two component vector process $(X(t), Y(t))$ where the first component $X(t)$ represents the state of Unit 1 with 0 and 1 respectively denoting the operable and failed state of Unit 1. The second component $Y(t)$ represents the number of failed units in the parallel system consisting of Unit 2 and Unit 3. The set of all possible states is given by

$$\{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2)\}$$

By the assumption 4 of the model it follows that the system cannot enter the state $(1,2)$

4.3 Reliability Analysis

Since the failure and the repair rates are constants $Z(t)$ is a Markov process with state space

$$S = \{(0,0),(0,1),(0,2),(1,0),(1,1)\}$$

We can write

$$S = S_U \cup S_D$$

where $S_U = \{(0,0),(0,1)\}$ and $S_D = \{(0,2),(1,0),(1,1)\}$. S_U corresponds to system up states and S_D that of down states. Define $\widetilde{P}_{ij,kl}(t) = Pr\{Z(t) = (k,l), Z(u) \neq (m,n) \forall u \in (0,t],$ $(m,n) \in S_D | Z(0) = (i,j)\}$ $(i,j),(k,l) \in S_U$ Using probabilistic arguments we derive the following equations:

$$\widetilde{P}_{00,00}(t) = e^{-(\lambda_1+2\lambda_2)t} + 2\lambda_2 e^{-(\lambda_1+2\lambda_2)t} © \widetilde{P}_{01,00}(t) \tag{1}$$

$$\widetilde{P}_{01,00}(t) = \mu_2 2\lambda_2 e^{-(\lambda_1+\lambda_2+\mu_2)t} © \widetilde{P}_{00,00}(t) \tag{2}$$

Talking Laplace transforms and then solving the equations (1) and (2) we get

$$\widetilde{P}^*_{00,00}(s) = \frac{s+\lambda_1+\lambda_2+\mu_2}{B1} \tag{3}$$

where

$$B1 = s^2 + s(2\lambda_1 + 3\lambda_2 + \mu_2) + \lambda_1(\lambda_1+\lambda_2+\mu_2) + 2\lambda_2(\lambda_1+\lambda_2)$$

Probabilistic arguments lead to the derivation of the following equations:

$$\widetilde{P}_{00,01}(t) = 2\lambda_2 e^{-(\lambda_1+2\lambda_2)t} © \widetilde{P}_{01,01}(t) \tag{4}$$

$$\widetilde{P}_{01,01}(t) = e^{-(\lambda_1+2\lambda_2+\mu_2)t} + \mu_2 e^{-(\lambda_1+2\lambda_2+\mu_2)t} © \widetilde{P}_{00,01}(t) \tag{5}$$

Talking Laplace transforms and then solving the equations (4) and (5) we get

$$\widetilde{P}^*_{00,01}(s) = \frac{s+2\lambda_2}{B1} \tag{6}$$

Define $R(t) = Pr\{$System is up in $(0,t] | X(0) = (0,0)\}$ Then $R(t)$ is the reliability of the system and is given by

$$R(t) = \widetilde{P}_{00,00}(t) + \widetilde{P}_{00,01}(t)$$

4.4 Mean time to System Failure

The mean time to system failure is given by

$$R^*(0) = \widetilde{P}^*_{00,00}(0) + \widetilde{P}^*_{00,01}(0)$$

$$= \frac{\lambda_1+3\lambda_2+\mu_2}{\lambda_1(\lambda_1+\lambda_2+\mu_2)+2\lambda_2(\lambda_1+\lambda_2)}$$

4.5 Availability Analysis

Define $P_{ij,kl}(t) = Pr\{Z(t) = (k,l)|Z(0) = (i,j)\} \quad (i,j),(k,l) \in S$

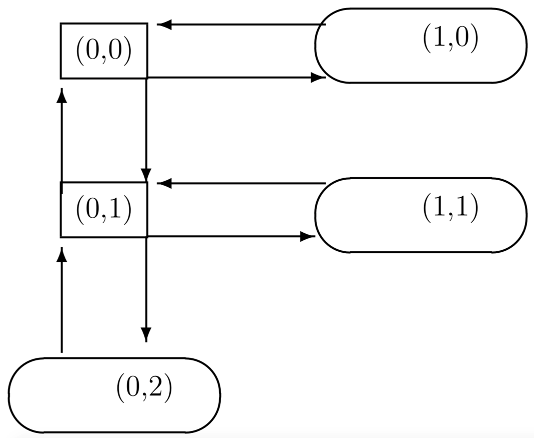

Figure 2

4.6 Steady state Analysis

Define

$$P_{ij} = \lim_{t \to \infty} P_{kl,ij}(t)$$

Using the principle of flow balance and the transition diagram given in Fig.2 we derive the following equations for P_{ij}

$$(\lambda_1 + 2\lambda_2) = \mu_1 P_{10} + \mu_2 P_{01}$$
$$(\lambda_1 + \lambda_2 + \mu_2)P_{01} = 2\lambda_2 P_{00} + \mu_1 P_{11} + \mu_2 P_{02}$$
$$\mu_1 P_{10} = \lambda_1 P_{00}$$
$$\mu_1 P_{11} = \lambda_1 P_{01}$$
$$\mu_2 P_{02} + \lambda_2 P_{01}$$

Solving these equation along with the equation

$$P_{00} + P_{01} + P_{10} + P_{11} + P_{02} = 1$$

we get $P_{00} = \mu_1 \mu_2^2 / B2$ and $P_{01} = 2\lambda_2 \mu_1 \mu_2 / B2$ where

$$B2 = \mu_1 \mu_2^2 + 2\lambda_2 \mu_1 \mu_2 + 2\lambda_1 \lambda_2 \mu_2 + 2\mu_1 \lambda_2^2$$

and the Steady state availability A_∞ is given by

$$A_\infty = P_{00} + P_{01} = \mu_1 \mu_2 (2\lambda_2 + \mu_2)/B2$$

5 Model 4 - MARKOV RENEWAL PROCESS

5.1 Assumptions

Assumption 3 of the previous model is given below : "There is a single repair facility and the repair rate of Unit 1 is μ_1 and that of Unit 2 and Unit 3 is μ_2. Preemptive priority is followed for the repair of Unit 1.

We modify this assumption as follows : 3. There is a single repair facility and the repair time distribution of Unit 1 is arbitrary and is taken as $g(.)$ The repair rate of Unit 2 and Unit 3 is μ_2. Preemptive priority is followed for the repair of Unit 1.

5.2 Analysis

Let $\tau_0 = 0, \tau_1, \cdots$ be the epochs at which the repair for Unit 1 commences. Define $Z_n = \{1, Y(\tau_n+)\}$ Then we observe that the stochastic process

$$(Z, T) = \{Z_n, \tau_n \, n = 0, 1, 2, \cdots\}$$

is a Markov Renewal Process on the state space $S = \{(1, 0), (1, 1)\}$ (see Cinlar (1985))The semi-Markov kernel is defined by

$$Q_{ij}(t) = \lim_{\Delta \to 0} P(Z_{n+1} = (1, j), t < \tau_{n+1} - \tau n \leq t + \Delta | Z_n = (1, i))/\Delta, i, j = 0, 1$$

To derive an expression for the semi-Markov kernel we define the following events and auxiliary functions :

E_{10}: repair for Unit 1 commences and both Unit2 and Unit 3 are operable.

E_{11}: repair for Unit 1 commences and Unit 2 (Unit 3) is operable and Unit 3 (Unit 2) is in the failed state.

E_{22}: Unit 1 is operable and Unit 2 (Unit 3) fails and Unit 3 (Unit 2) is in the failed state.

Define $f_{00,10}(t) = \lim_{\Delta \to 0} P(E_{10}\text{in } (t, t + \Delta), N(E_{10}, t) = 0, N(E_{11}, t) = 0|$
$Z(0) = (0, 0))/\Delta$ $f_{00,11}(t) = \lim_{\Delta \to 0} P(E_{11}\text{in } (t, t + \Delta), N(E_{10}, t) = 0, N(E_{11}, t) = 0|$
$Z(0) = (0, 0))/\Delta$ $f_{01,10}(t) = \lim_{\Delta \to 0} P(E_{10}\text{in } (t, t + \Delta), N(E_{10}, t) = 0, N(E_{11}, t) = 0|$
$Z(0) = (0, 1))/\Delta$ $f_{00,11}(t) = \lim_{\Delta \to 0} P(E_{11}\text{in } (t, t + \Delta), N(E_{10}, t) = 0, N(E_{11}, t) = 0|$
$Z(0) = (0, 1))/\Delta$

Using probabilistic arguments we derive the following equations:

$$f_{00,10}(t) = \lambda_1 e^{-(2\lambda_2+\lambda_1)t} + 2\lambda_2 e^{-(2\lambda_2+\lambda_1)t} © f_{01,10}(t)$$

$$f_{00,11}(t) = \lambda_1 e^{-(2\lambda_2+\lambda_1)t} © f_{10,11(t)} + 2\lambda_2 e^{-(2\lambda_2+\lambda_1)t} © f_{01,10}(t)$$

$$f_{01,10}(t) = \lambda_2 e^{-(\lambda_1+\lambda_2+\mu_2)t} © \mu_2 e^{-\mu_2 t} © f_{01,10}(t) + \mu_2 e^{-(\lambda_1+\lambda_2+\mu_2)t} © f_{00,10}(t)$$

$f_{01,11}(t) = \lambda_1 e^{-(\lambda_1+\lambda_2+\mu_2)t}$
$+\lambda_2 e^{-(\lambda_1+\lambda_2+\mu_2)t} © \mu_2 e^{-\mu_2 t} © f_{01,11}(t) + \mu_2 e^{-(\lambda_1+\lambda_2+\mu_2)t} © f_{00,11}(t)$

5.3 The semi Markov kernel

The semi-Markov kernel is given by

$$Q_{ij}(t) = g(t)©f_{0i,1j}(t), \quad i,j = 0,1$$

Using the Theory of Markov Renewal Process we obtain an expression for the Markov Renewal function. The stochastic Process $\{Z(t); t \geq 0\}$ is a Semi-Regenerative (Cinlar(1985)) Process with the Markov Renewal Process (Z,T) embedded in it. The distribution of $Z(t)$ is defined by
$P(ij,kl,t) = P(Z(t) = (k,l)|E_{ij}$ at $t = 0)$, $(i,j),(k,l) \in S$ The availability of the system is given by

$$A(t) = \sum_{ij \in S_U} P(00,ij,t)$$

In Model 2 the assumptions are 1. The life time (LT)of Unit 1 is a random variable X with an arbitrary distribution having pdf $f(.)$ .Unit 2 and Unit 3 are statistically independent and identical and have the same failure rate λ 2. When Unit 1 fails it is instantaneously taken up for repair and the repair time (RT) is a random variable Y with an arbitrary distribution having pdf $g(.)$. When Unit2 and Unit 3 fail they are instantaneously replaced with a similar new unit. E_o and E_1 respectively denote the events that Unit 1 just begins to operate and the repair for it just commences. Also let $\tau_0 = 0, \tau_1, \cdots$ be the epochs at which any of the events E_0 or E_1 occur. Define $Z_n = \{Z(\tau_n+)\}$ Then we observe that the stochastic process

$$(Z,T) = \{Z_n, \tau_n \, n = 0,1,2,\cdots\}$$

is a Markov Renewal Process on the state space $S = \{0,1\}$. The stochastic Process $\{Z(t); t \geq 0\}$ is a Semi-Markov Process with the Markov Renewal Process (Z,T) embeded in it. The distribution of $Z(t)$ is defined by $P(i,j,t) = P(Z(t) = j|E_i$ at $t = 0)$, $i,j \in S$ and can be easily obtained . The availability of the system is given by

$$A_i(t) = P(0,i,t)$$

Model	Unit 1	Unit2	Unit3	Stochastic Process
1	Replacement upon failure	Replacement upon failure	Replacement upon failure	Renewal Process
2	LT: G RT: G	Replacement upon failure	Replacement upon failure	Alternating Renewal Process
3	FR: λ_1 RR:μ_1	FR:λ_2 RR:μ_2	FR:λ_2 RR:μ_2	Markov Process
4	LT: G RT: G	FR:λ_2 RR:μ_2	FR:λ_2 RR:μ_2	Markov Renewal Process

LT- Life time RT - Repair time FR- Failure rate RR-Repair rate
Model 2 Semi-Markov Process Model 4 Semi-Regenerative Process

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Dr VSS Yadavalli is a Professor and Head of Department of Industrial & Systems Engineering, University of Pretoria. Professor Yadavalli has published over 150 research paper mainly in the areas of Reliability and inventory modelling in various international journals like, International Journal of Production Economics, International Journal of Production Research, International Journal of Systems Science, IEEE Transactions on Reliability, Stochastic Analysis & Applications, Applied Mathematics & Computation, Annals of Operations Research, Computers & Industrial Engineering etc. Prof Yadavalli is an NRF (South Africa) rated scientist and attracted several research projects. Prof Yadavalli is serving several editorial boards of various international journals. He supervised several M and D students locally and abroad. Professor Yadavalli was the past President of the Operations Research Society of South Africa. He is a 'Fellow' of the South African Statistical Association. He received a 'Distinguished Educator Award 'from Industrial Engineering and Operations Management Society in 2015.