Proposition 2.5 (Fiedler 1973). *The Laplacian eigenvalues of the Cartesian product G* x *H are precisely all sums*

$$\lambda_i(G) + \lambda_j(H), i = 1,\ldots,|G|, \qquad j = 1,\ldots,|H|.$$

In particular,

$$\lambda_2(G \times H) = \min\{\lambda_2(G), \lambda_2(H)\} \text{ and } \lambda_\infty(G\ x\ H) = \lambda_\infty(G) + \lambda_\infty(H)$$

for nonnegatively weighted graphs G and H.

Let us mention some known bounds on λ_∞. First (Anderson and Morley 1985)

$$\lambda_\infty \leq \max\ \{d(u)\ +\ d(u)\ |uv\ \in\ E(G)\}$$

where $d(u)$ is the degree of the vertex u (the sum of the weights of edges incident with u in the weighted case). If G is connected then, in the above inequality, there is equality if and only if G is bipartite semiregular. Also (Kel'mans 1967), $\lambda_\infty \leq n$ with equality if and only if the complement of G is not connected. Let us mention two other relations about λ_∞:

$$\sum_{i=1}^{n} \lambda_i = 2|E(G)| = \sum_{v \in V} d(v)$$

and (Fiedler 1973)

$$\lambda_\infty \geq \frac{n}{n-1} \max\ \{d(v)|\ v\ \in\ V(G)\}$$

The Laplacian spectrum can directly be obtained from the adjacency spectrum in case G is an (unweighted) d-regular graph. Let A be the adjacency matrix of G and $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$ its eigenvalues. Then $d - \mu_n \leq \ldots \leq d - \mu_1$ are the Laplacian eigenvalues of G (Cvetkovic 1979). We will use this relation in the next two sections, since most graphs considered there will be regular unweighted. The adjacency spectrum has been more studied so far, and all the facts we need may be found in (Cvetkovic 1979).

3. MAX-CUT AND EIGENVALUES IN SPECIAL CLASSES

In this section we examine two classes of graphs for for which the value of max-cut is known, and where the eigenvalue upper bound is not optimum. For Kneser graphs *K(n, r)* the eigenvalue upper bound agrees with an upper bound obtained from the size of maximum clique, and it is quite satisfactory. On the contrary, the boundis poor for some circulants. We exhibit a sequence $\{G_n\}$ of graphs of order n where $|E(G_n)| - MC(G_n)$ is increasing while $|E(G_n)| - \frac{1}{4}\lambda_\infty n$ is bounded.

Kneser graphs. Kneser graph *K(n,* r) is the graph whose vertex set is formed by all r-subsets of an n-set, and two r-subsets form an edge if they are disjoint. We will consider only case $r = 2$. Since $K(n, 2) = \overline{L(K_n)}$,the complement of the line graph of K_n, we have

$$\lambda_\infty K(n,2) = \lambda_\infty \overline{L(K_n)} = \binom{n}{2} - n.$$

The exact value of max-cut in $K(n, 2)$ has been found in (Poljak and Tuza 1987). The max-cut is formed by $\delta(S)$ for $S = \{\{i, j\} \mid \min(i,j) \leq P_n\}$ where $P_n = \left[(2-\sqrt{2})\, {}^{n}\!/_{2}\right]$. It is more informative to look at the *bipartite density* instead of the value of max-cut. (The bipartite density of a graph G is defined as the ratio $MC(G)/|E(G)|$.) Since each edge belongs to the same number of maximum cliques, the bipartite density of a Kneser graph is bounded by the bipartite density of its maximum clique. The upper bound on the bipartite density of $K(n, 2)$ derived from λ_∞ is

$$\frac{1}{2}+\frac{1}{n-2}$$

It is compared with the bound obtained from the size of maximum clique in the following table.

Kneser graph	bound from max. clique	eigenvalue u.b.
$K(4n, 2)$	$\frac{1}{2}+\frac{1}{4n-2}$	$\frac{1}{2}+\frac{1}{4n-2}$
$K(4n+1, 2)$	$\frac{1}{2}+\frac{1}{4n-2}$	$\frac{1}{2}+\frac{1}{4n-1}$
$K(4n+2, 2)$	$\frac{1}{2}+\frac{1}{4n-2}$	$\frac{1}{2}+\frac{1}{4n}$
$K(4n+3, 2)$	$\frac{1}{2}+\frac{1}{4n-2}$	$\frac{1}{2}+\frac{1}{4n+1}$

For a general Kneser graph $K(n, r)$, the maximum eigenvalue

$$\lambda_\infty\big(K(n,r)\big) = \binom{n-r}{r}\binom{n-r-1}{r-1}$$

has been determined by (Lovász 1979). This gives an upper bound $1/2(1 + r/(n-r))$ on the bipartite density of $K(n, r)$, and the bound is very close to that derived from the size of the maximum clique. The exact solution is known only for $n \leq (4.3 + o(1))\, r$ (Poljak and Tuza 1987).

Circulants. Let $w = (w_1, w_2, \ldots, w_{n-1})$ be a real vector such that $w_i = w_{n-i}$, for all i. The w-circulant is the weighted graph C_w with vertices 0, 1, ..., $n-1$ and the weights w_{j-1} on the edge ij , $i < j$. For example, cycles are w-circulants with $w_1 = w_{n-1} = 1$ and $w_i = 0$ otherwise.

Denote by $d(w) := \sum_{i=1}^{n-1} w_i$. Let $w^{(i)}$ be the vector with i-th and $(n-i)$-th entry equal 1 and equal 0 otherwise. Denote by A_i the n x n matrix with entries 0 except $(A_i)_{jk} = 1$ if $k - J = i \pmod{n}$. It is well known (Cvetkovic 1979) that the eigenvalues of A_1 are the n-th roots of unity, and that $A_1 = A_1^i$. We have

Lemma 3.1. *The Laplacian matrix of C_w is given by*

$$Q(C_w) = d(w)I - \sum_{j=1}^{n-1} w_j A_1^j$$

and its spectrum consists of numbers

$$v_p = d(w)\sum_{j=1}^{n-1} w_j \exp(\frac{2\pi i}{n} jp), \qquad p = 0, 1, \ldots, n-1$$

.
where i is the imaginary unit.

Notice that $v_0 = 0$ and some v_p may have the same value. We will consider only a subclass of circulants in the sequel. We denote by $C_{n,r}$ the circulant given by $w_1 = w_{n-1} = w_r = w_{n-r} = 1$ and $w_i = 0$ otherwise. Mention that, for $r < n/2$, $C_{n,r}$ is a 4-regular graph consisting of a cycle of length n and all chords connecting vertices ofdistance r on the cycle. The exact value of max-cut in $C_{n,r}$ has been found in (Poljak and Turzik 1986) [PT2].

Proposition 3.2 (Poljak and Turzik 1986) *The max-cut of* $C_{n,r}$, $r < n/2$, *is given by* $MC(C_{n,r}) = 2n - d$ *where* $d = \min(p + |tn - pr|)$ *and the minimum is taken over pairs p, t of nonnegative integers satisfying* $p = n$ (mod 2) *and* $t \neq r$ (mod 2).

Mention that one can compute d by examining all values of t = 0, 1, ..., n and taking the best p for each t. The case $r = n/2$ is not so interesting since the circulant $C_{2r,r}$ is either bipartite or becomes bipartite after deleting two edges. It follows from Lemma 3.1 that

(6) $\lambda_\infty(C_{n,r}) = 4 - 2 \min_{0 \le p \le n-1} \left(cos\frac{2\pi p}{n} + cos\frac{2\pi pr}{n} \right)$

for $r < n/2$. Using (6) and Proposition 3.2 we compare the eigenvalue upper bound with the exact value of $MC(C_{n,r})$. The results, for some small values of n and r, are in the following table. We excluded $C_{5,2}$ (K_5), and the pairs n even, r odd since $C_{n,r}$ is bipartite for such parameters. It will be shown in the next section that the upper bound is exact for complete and bipartite regular graphs.

n	r	$MC(C_{n,r})$	$\lambda_\infty . n/4$
6	2	8	9
7	2	10	10.9
7	3	10	10.9
8	2	12	12 exact value
9	2	12	13.5
9	3	14	15.5
9	4	12	13.5
10	2	14	15.6
10	4	16	18.1
11	2	16	16.5
11	3	18	19.8
11	4	18	19.8
11	5	16	16.5
12	2	18	18 exact value
12	4	18	20.2
13	2	18	20.2

13	3	22	24.2
13	4	22	24.2
13	5	20	21.6
13	6	18	20.2
14	2	20	21.8
14	4	22	24.7
14	6	24	26.6
15	2	22	23.1
15	3	26	28.4
15	4	24	27.1
15	5	24	26.1
15	6	24	27.1
15	7	22	23.1
16	2	24	24.7
16	4	28	29.6

Since it seems difficult to find an explicit formula for $\lambda_\infty(C_{n,r})$, we will investigate two special classes.

Circulants $C_{n,2}$. Using Proposition 3.2 we get

$$MC(C_{4k,2}) = 6k \text{ , and } MC(C_{4k+i,2}) = 6k + 2(i - 1),\ i = 1, 2, 3 .$$

For k large we have $\lambda_\infty C_{4k,2} = 6.25$, which gives an upper bound $6.25k$ (while the actual value is $6k$).

Circulants $C_{r2+1,r}$, r even. Lising Proposition 3.2 we get $MC(C_{r2+1,r}) = 2n - 2r = 2(r2 - r + 1)$. The maximum eigenvalue can be estimated

$$\lambda_\infty\ (C_{r2+1,r}) = 8 - c^{r-2} + O(r^{-3})\ (c \sim 2\pi^2).$$

Hence the eigenvalue upper bound tends to $2n - \frac{\pi^2}{2}$.

Ramanujan graphs are r-regular graphs for which

$$\lambda_2(G) \geq r - 2\sqrt{(r-1)} \text{ and } \lambda_\infty \leq r + 2\sqrt{(r-1)}.$$

This interesting class of graphs was introduced by Lubotzky, Phillips and Sarnak (Lubotzky et al. 1988)[LPS], and for any $r = p + 1$, where p is a prime congruent to 1 mod 4, an infinite family was constructed. We have $MC(G) \leq \frac{1}{4}nr + \frac{1}{2}n\sqrt{(r-1)}$ for a Ramanujan graph G.

4. EXACT GRAPHS

The eigenvalue upper bound of Theorem 2.2 can be tight only in case that we have large cuts separating two large sets of vertices (each close to half of the vertices). Examples of such graphs are complete graphs and their Cartesian products, or tensor (categorical) products.
In this section we describe some classes for which the upper bound is best possible.
For simplicity, we restrict ourselves to graphs ofeven order only. Let us call a graph *G exact* if $MC(G) = \lambda_\infty n/4$. We show that the following graphs are exact (with possible restriction on even parity of some parameters): complete graphs and their categorical and cartesian

products, bipartite regular graphs, line graphs of semiregular bipartite graphs, line graph $L(K_{4k+1})$, complement *of* $L(K_{m,n})$. We also show thatexact graphs are closed under the cartesian product. The maximal cuts in these graphs are easily found, and Theorem 2.2 provides a proof of their optimality. The used facts on λ_∞ may be found in (Cvetkovic 1979) (cf. remark in the end of Section 2).

Proposition 4.1. *The cartesian product G* x *H of two exact graphs is exact.*

Proof. We have $\lambda_\infty(G \times H) = \lambda_\infty(G) + \lambda_\infty(H)$ by Proposition 2.5. Conversely, let $\delta(V_0)$ and $\delta(W_0)$ be the maximal cut of G and H, respectively. Then $|V_0|=\frac{1}{2}|V(G)|$, $W_0=\frac{1}{2}|V(H)|$, and $\delta(V_0 \times W_0 \cup (V(G) \setminus V_0) \times (V(H) \setminus W_0))$ is the maximal cut in the product.

Complete graphs. We have $\lambda_\infty K_n = n$. The max-cut in K_n is obviously $\left\lceil\frac{n}{2}\right\rceil\left\lfloor\frac{n}{2}\right\rfloor$ which agrees with the upper bound. Hence the cartesian product ofcomplete graphs of even order is exact. We show that also complements of these products are exact.

Categorical product. $G \otimes H$ of G and H is the graph with vertex set $V(G) \times V(H)$ and edges (u, v) (u', v') if $uu' \in E(G)$ and $vv' \in E(H)$.

The product $K_m \otimes K_n$ of complete graphs equals $\overline{K_m \times K_n}$, the complement of their cartesian product. Since

$$\lambda_\infty \overline{G \times H} = mn - \lambda_2(G \times H) = mn - \min\{\lambda_2(G), \lambda_2(H)\},$$

where $n = |G|$ and $m = |H|$, we have $\lambda_\infty(K_n \otimes K_m) = mn - \min(m, n)$.

Proposition 4.2. *Let* $n \leq m$, *n even. Then* $K_n \otimes K_m$ *is exact.*

Proof. The maximal cut is $\delta(\{1,\ldots,\frac{1}{2}n\} \times \{1,\ldots,m\})$.

The results easily generalize to products of greater number of complete graphs. In particular, the max-cut in the complement of d-dimensional cartesian cube is $2^{d-1}(2^{d-1}-1)$.

Bipartite regular graphs are exact, since $\lambda_\infty(G) = 2r$ for an r-regular bipartite graph G.

Line graphs of bipartite graphs and their complements. A bipartite graph G is *(r, s)-semiregular* if r and s are the degrees in either bipartite class. If $r \neq 1$ and $s \neq 1$, we have

$$\lambda_\infty L(G) = r + s \text{ , and}$$
$$\lambda_\infty \overline{L(G)} = |E(G)| - r - s + \lambda_{n-1}(G), \quad n = |V(G)|.$$

In particular, for $m, n \neq 1$, we have

$$\lambda_\infty \overline{L(K_{m,n})} = mn - \min(m, n).$$

Proposition 4.3. *Let* ***G*** *be a bipartite* (r, s)*-semiregular graph where both r and s are even. Then* $L(G)$ *is exact.*

Proof. The edge set $E(G)$ can be decomposed into two $(\frac{1}{2}r, \frac{1}{2}s)$-semiregular subgraphs which form the optimum bipartition of $L(G)$. The eixstence ofsuch decomposition of $E(G)$ is well known.

Proposition 4.4. *Let* $n \leq m$, *n even. Then* $\overline{L(K_{m,n})}$ *is exact.*

Proof. We have $\lambda_\infty = n(m-1)$. The max-cut is obtained by $\delta(\{ij \mid i = 1,\ldots,1/2\, n, j = 1,\ldots,m\})$.

Line graphs of complete graphs. We have $\lambda_\infty L(K_n) = 2(n-1)$.

Proposition 4.5. $L(K_{4r+1})$ *is exact.*

Proof. It is well-known that K_{4r+1} has a $2r$-regular factor, which is the maximal cut in the line graph.

Let us remark that also the circulants $C_{8,2}$ and $C_{12,2}$ are exact.

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