The Completeness of p-Summable Sequences as 2-Normed Spaces

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Abstract

The norm space is the pairs of vector space with defined norm in the vector space. An example of vector space is ℓ^p. Vector space ℓ^p is a space that contains all real number sequences that satisfy the sums of all elements in ℓ^p normed space. Furthermore at ℓ^p can be defined as a norm-2 $\|\cdot,\cdot\|_p$, that is equal to the sum of all 2 dimensional matrix determinant in norms in ℓ^p, such that $(\ell^p, \|\cdot,\cdot\|_p)$ is a 2-normed space. Result of this study is seeking at the completeness of 2-normed space $(\ell^p, \|\cdot,\cdot\|_p)$by utilizing the completeness of norm space $(\ell^p, \|\cdot\|_p)$.

Keywords
p-summable, norm space, Banach space

1. Introduction

Normed space is a vector space which equipped with a norm as a function(Ganzburg 2010; González and Martínez-Abejón Antonio 2012). Geometrically, norm is a simple tool to obtains a vector's length(Idris and Gunawan 2016). In real life, to calculates an area's large(Li et al. 2002) which is stretched by 2 vectors and need more complicated measurement tool. To answer that problem, Gähler (1964) defines a measurement tool for calculates an area's large (Idris and Gunawan 2016).

Vector space p-summable(ℓ^p) is a space that contains sequences of real numbers that satisfied $\sum_{k=1}^{\infty}|x_k|^p < \infty$ (Chu 2007). ℓ^p vector space with $\|\cdot\|_p$ is a normed space. Then note that $(\ell^p, \|\cdot,\cdot\|_p)$ is Banach space(Marciszewski and Pol 2009). Otherwise, ℓ^p can be seen as a 2-normed space, with norm-2 in ℓ^p is defines below.

$$\|x,y\|_p = \left[\frac{1}{2}\sum_k\sum_l \left|det\begin{pmatrix} x_k & x_l \\ y_k & y_k\end{pmatrix}\right|^p\right]^{1/p}$$

This study about norm-2 space(Li et al. 2002) and its completeness will be limited by using ℓ^p vector space along with norm-2 that fulfills it.

2. Material and Methods

2.1 Norm Space

Vector space is scope that will be basis of this research. Therefore, then definition of vector space below will be explained(Idris and Gunawan 2016).

Definition 1. Let X is a nonempty set and $\mathbb{R}$ is set of real numbers. Pair of $(X, \mathbb{R}, +, \cdot)$ with $x, y, z \in X$, that satisfy

1. If for any $x, y \in X$, then $x + y \in X$,
2. $(x + y) + z = x + (y + z)$, for any $x, y, z \in X$,
3. $x + y = y + x$, for any $x, y \in X$,
4. There is $0 \in X$ such that $x + 0 = 0 + x = x$, for any $x \in X$,
5. If for any $x \in X$ and $\alpha \in \mathbb{R}$, then $\alpha \cdot x \in X$,
6. For any $x \in X$ there is $-x \in X$ such that $x + (-x) = 0$,
7. $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$, for any $x \in X$ and $\alpha, \beta \in \mathbb{R}$
8. $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$, for any $x, y \in X$ and $\alpha \in \mathbb{R}$
9. $\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$, for any $x \in X$ and $\alpha, \beta \in \mathbb{R}$
10. $1 \cdot x = x$, for any $x \in X$,

notated by X is called real vector space.

Example 1. $\boldsymbol{\ell^p}$ space is a real vector space.

For any real vector space, it can defines a function that describes below (Chu et al. 2008; Šemrl 2008).

Definition 2. Let X is vector space in $\mathbb{R}$. Norm in X defined as function $\|\cdot\|: X \to \mathbb{R}$ that satisfy

1. $\|x\| \geq 0$, for any $x \in X$ and $\|x\| = 0 \leftrightarrow x = 0$,
2. $\|\alpha x\| = |\alpha|\|x\|$, for any $x \in X$ and $\alpha \in \mathbb{R}$,
3. $\|x + y\| \leq \|x\| + \|y\|$, for any $x, y \in X$.

Pair of $(X, \|\cdot\|)$ with $\|\cdot\|$ is a norm from vector space X is called norm space.

Example 2. ℓ^pspace, $1 \leq p < \infty$ is a space that fullfil all real numbers sequence (x_n) which satisfies $[\sum_i |x_i|^p] < \infty$, with function

$$\|x\|_p = \left(\sum_{i=1}^{\infty}|x_i|^p\right)^{\frac{1}{p}}.$$

In [4] was explained some sequence definitions that will help this paper.

Definition 3. A sequences (x_n) is a function that paired on natural numbers set $\mathbb{N} = \{1,2,\cdots\}$ to real numbers set.

Let $(x_n): \mathbb{N} \to \mathbb{R}$ is a sequence. Convergence of any sequence can be notes as

Definition 4. A sequence (x_n) in X is said convergent to $x \in X$, if for any $\varepsilon > 0$ there is $K(\varepsilon) \in \mathbb{N}$ such that for all $n \geq K$, then satisfy $\|x_n - x\| < \varepsilon$.

Next definition will explains about Cauchy sequence

Definition 5. A sequence (x_n) is a Cauchy sequence if for any $\varepsilon > 0$ there is $H(\varepsilon) \in \mathbb{N}$ such that for all $n, m \geq H(\varepsilon)$ then $\|x_n - x_m\| < \varepsilon$.

A convergent sequence (x_n) in X have a relation with Cauchy sequence that explained as follow.

Lemma 1. If (x_n) is a convergent sequence in X space, then (x_n) is a Cauchy sequence.

From a sequence of norm space, can set up a new definition of the complete norm space as follow(Idris and Gunawan 2016).

Definition 6. Let X is a norm space. If any Cauchy sequence in X is convergent to $x \in X$, then X is a complete norm space or called Banach space.

Example 3. $(\ell^p, \|\cdot\|_p)$ with $\|x\|_p \coloneqq [\sum_k |x_k|^p]^{1/p}$ is a Banach space.

Next is how to know equivalence from two norms will be explained below.

Definition 7. Norm $\|\cdot\|$ is equivalent to $\|\cdot\|^*$ if there are two positive integers α, β such that for all $x \in X$ then

$$\alpha\|x\|^* \leq \|x\| \leq \beta\|x\|^*$$

Inequality from Definition 7 will be used as a connector to problem in this paper. And then there is triangle inequality of ℓ^p that will explained below.

Definition 8. Let $(\ell^p, \|\cdot\|_p)$ is a norm space, if vectors $x, y \in (\ell^p, \|\cdot\|_p)$ and $p \geq 1$, then

$$\left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{\frac{1}{p}}$$

2.2 2-Normed Space

In vector space X, can defines a function that map vector space X^2 to real numbers $\mathbb{R}$. Note the following definition (Šemrl 2008).

Definition 9. Let X is a vector space. Function $\|\cdot,\cdot\|: X^2 \to \mathbb{R}$ that satisfy

1. $\|x, y\| = 0$ if and only if x and y is linearly independent for any $x, y \in X^2$,
2. $\|x, y\| = \|y, x\|$, for any $x, y \in X^2$,
3. $\|\alpha x, y\| = |\alpha|\|x, y\|$, for any $x, y \in X^2$ and $\alpha \in \mathbb{R}$,
4. $\|x + y, z\| \leq \|x, z\| + \|y, z\|$, for any $x, y, z \in X^2$,

is norm-2 in X. Pair of $(X, \|\cdot,\cdot\|)$ is called 2-normed space.

In norm vector X fulfill with norm-2, can defines sequence convergence and Cauchy sequence below.

Definition 10. A sequence $(\boldsymbol{x_n})$ in $(\boldsymbol{X}, \|\cdot,\cdot\|)$ is called convergent to $\boldsymbol{x} \in \boldsymbol{X}$ if for all $\boldsymbol{y} \in \boldsymbol{X}$ is satisfy

$$\|x_n - x, y\| \to 0, \text{ dengan } n \to \infty$$

Definition 11. A sequence (x_n) in 2-normed space $(\ell^p, \|\cdot,\cdot\|_p)$ is called Cauchy sequence if for all $y \in X$ is satisfy

$$\|x_m - x_n, y\| \to 0, \text{ dengan } m, n \to \infty.$$

Connection about both of that given by definition below.

Definition 12. Let X is a 2-normed space. If all Cauchy sequence on X convergent to x in X, (Idris and Gunawan 2016) then X is called complete 2-normed space or 2-Banach space.

Example 3. $\boldsymbol{\ell^p}$ vector space with $1 \le p < \infty$, fulfill with this function

$$\|x, y\|_p = \left[\frac{1}{2}\sum_k \sum_l \left|det\begin{pmatrix} x_k & x_l \\ y_k & y_l \end{pmatrix}\right|^p\right]^{\frac{1}{p}},$$

is 2-normed space.

2.3 Completeness of ℓ^p as 2-Normed Space

A vector space ℓ^p with $1 \le p < \infty$ and $x = (x_k), y = (y_k) \in \ell^p$, with function

$$\|x, y\|_p = \left[\frac{1}{2}\sum_k \sum_l \left|det\begin{pmatrix} x_k & x_l \\ y_k & y_l \end{pmatrix}\right|^p\right]^{\frac{1}{p}}$$

is 2-normed space. As known that $(\ell^p, \|\cdot\|_p)$ is Banach space. From that all Cauchy sequence in $(\ell^p, \|\cdot\|_p)$ is a convergent sequence to $x \in \ell^p$. If sequence (x_n) in $(\ell^p, \|\cdot,\cdot\|_p)$ is convergent to $x \in X$,

$$\lim_{n\to\infty} \|x_n - x, y\| = 0,$$

then sequence (x_n) is a Cauchy sequence is $(\ell^p, \|\cdot\|_p)$. That is shown below.

$$\lim_{n\to\infty} \|x_n - x_m, y\| \le \lim_{n\to\infty} \|x_n - x, y\| + \lim_{n\to\infty} \|x_m - x, y\| \to 0 + 0 = 0$$

Shown by Gunawan, defines a norm form 2-norm below.
Let $e_1 \coloneqq (1,0,0,\cdots) \in \ell^p$ and $e_2 \coloneqq (0,1,0,\cdots) \in \ell^p$, this function

$$\|x\| \coloneqq \left[\|x, e_1\|_p^p + \|x, e_2\|_p^p\right]^{1/p}$$

is a norm in ℓ^p. Generally, if $\{a, b\}$ is an independent set in ℓ^p, then

$$\|x\|_0 \coloneqq \left[\|x, a\|_p^p - \|x, b\|_p^p\right]^{1/p}$$

is norm in ℓ^p.

From definition of $\|\cdot\|_0$ will show that if (x_n) is Cauchy sequence in $(\ell^p, \|\cdot,\cdot\|_p)$, then (x_n) is Cauchy sequence in $(\ell^p, \|\cdot\|_0)$. With substituting x with $x_n - x_m$ in $\|x\|_0$. Note that

$$\|x\|_0 = \left(\|x, a\|_p^p - \|x, b\|_p^p\right)^{\frac{1}{p}}$$

$$\|x_n - x_m\|_0 = \left(\|x_n - x_m, a\|_p^p + \|x_n - x_m, b\|_p^p\right)^{\frac{1}{p}} \to (0 + 0)^{1/p} = 0$$

Lemma 1. For all $x, y \in \ell^p$ is obtains $\|x, y\|_p \le 2^{1-\frac{1}{p}} \|x\|_p \|y\|_p$.

Lemma 2. For all $x, y, z \in \ell^p$ is obtains $\|x\|_p \|y, z\|_p \le 2\|y\|_p \|x, z\|_p + 2\|z\|_p \|x, y\|_p$.

Theorem 1. For a independent set $\{a, b\}$ in ℓ^p, norm $\|\cdot\|_0$ is equivalent to norm $\|\cdot\|_p$.

Proof. For all $x \in \ell^p$ is obtains $\|x, a\|_p \le 2^{1-1/p} \|x\|_p \|a\|_p$ and $\|x, b\|_p \le 2^{1-1/p} \|x\|_p \|b\|_p$, based on Lemma 1. such that

$$\|x\|_0 = \left[\|x, a\|_p^p + \|x, b\|_p^p\right]^{1/p} \le \left[\|x, a\|_p^p\right]^{\frac{1}{p}} + \left[\|x, b\|_p^p\right]^{\frac{1}{p}}$$
$$\le 2^{1-\frac{1}{p}} \left[\|x\|_p^p \|a\|_p^p\right]^{\frac{1}{p}} + 2^{1-1/p} \left[\|x\|_p^p \|b\|_p^p\right]^{1/p}$$
$$= 2^{1-1/p} \left[\|a\|_p^p + \|b\|_p^p\right]^{1/p} \|x\|_p.$$

While from Lemma 2. obtains $\|x\|_p \|a, b\|_p \le 2\|a\|_p \|x, b\|_p + \|b\|_p \|x, a\|_p$ dan $\|x\|_p \|b, a\|_p \le 2\|b\|_p \|x, a\|_p + \|a\|_p \|x, b\|_p$, such that

$$2\|x\|_p\|a,b\|_p \leq 3\|a\|_p\|x,b\|_p + 3\|b\|_p\|x,a\|_p.$$

Because $\|x,a\|_p \leq 2^{1-\frac{1}{p}}\|x\|_p\|a\|_p = \|x\|_0$ and $\|x,b\|_p \leq 2^{1-\frac{1}{p}}\|x\|_p\|b\|_p = \|x\|_0$, it results

$$2\|x\|_p\|a,b\|_p \leq 3\|a\|_p\|x,b\|_p + 3\|b\|_p\|x,a\|_p$$
$$\leq 3\big(\|a\|_p + \|b\|_p\big)\|x\|_0$$

or

$$\frac{2\|a,b\|_p}{3\big(\|a\|_p + \|b\|_p\big)}\|x\|_p \leq \|x\|_0.$$

That show that $\|\cdot\|_0$ is equivalent to $\|\cdot\|_p$ ■

Corollary 1. $(\ell^p, \|\cdot\|_0)$ space is Banach space.

From Definition 6. about Banach space and with explanation above, obtains that $(\boldsymbol{\ell^p}, \|\cdot\|_0)$ space is Banach space. After that, form Lemma 1. is obtain

$$\|x,y\|_p \leq 2^{1-\frac{1}{p}}\|x\|_p\|y\|_p$$
$$\|x_n - x, y\|_p \leq 2^{1-1/p}\|x_n - x\|_p\|y\|_p$$

Since $y \in \ell^p$, then $\sum_k |y_k|^p < \infty$, such that $\|y\|_p$ is finite. As result, if (x_n) is a convergent sequence in $\big(\ell^p, |\cdot|_p\big)$, then $2^{1-1/p}\|x_n - x\|_p\|y\|_p \to 0$. From that, the results can be obtained from the following theorem.

Theorem 2. In ℓ^p space, if a sequence (x_n) is convergent to $x \in \ell^p$ in $\|\cdot,\cdot\|_p$ term, then this sequence will convergent to x in $\|\cdot\|_p$. Similar to that, if sequence (x_n) is a Cauchy sequence in $\|\cdot,\cdot\|_p$ term, then this sequence is Cauchy sequence in $\|\cdot\|_p$.

Proof. (1) Let $\{a,b\}$ is independent set in ℓ^p and $\|\cdot\|_0$ is defined. If (x_n) is convergent to x in $\|\cdot,\cdot\|_p$, then

$$\|x_n - x, a\| \to 0, \text{ for } n \to \infty$$

and

$$\|x_n - x, b\| \to 0, \text{ for } n \to \infty.$$

This resulted

$$\|x_n - x\|_0 \to 0, \text{ for } n \to \infty$$

its mean (x_n) is convergent to x in $\|\cdot\|_0$. From Theorem 1., it has been obtain that (x_n) is convergent to x in $\|\cdot\|_p$.
(2) Let $\{a,b\}$ is independent set in ℓ^p and $\|\cdot\|_0$ is defined. If (x_n) is a Cauchy sequence in $\|\cdot,\cdot\|_p$, then

$$\|x_m - x_n, a\| \to 0, \text{ for } m,n \to \infty$$

And

$$\|x_m - x_n, b\| \to 0, \text{ for } m,n \to \infty.$$

This resulted

$$\|x_m - x_n\|_0 \to 0, \text{ for } n \to \infty$$

its mean (x_n) is Cauchy sequence in $\|\cdot\|_0$. From Theorem 1., it has been obtain that (x_n) is Cauchy sequence in $\|\cdot\|_p$. ■

Theorem 2. resulted that $\big(\ell^p, \|\cdot,\cdot\|_p\big)$ is a 2-Banach space. This is explained in the following corollary

Corollary 2. $\big(\ell^p, \|\cdot,\cdot\|_p\big)$ is a 2-Banach space.

Proof. let (x_n) is a Cauchy sequence in ℓ^p with $\|\cdot,\cdot\|_p$ term. Then with Theorem 2. has obtained that (x_n) is a Cauchy sequence in $\|\cdot\|_p$. Since $\big(\ell^p, \|\cdot\|_p\big)$ is a Banach space, then for all Cauchy sequences (x_n) in ℓ^p is convergent to $x \in \ell^p$ in $\|\cdot\|_p$. From Lemma 2. (x_n) is convergent in $\|\cdot,\cdot\|_p$, such that $\big(\ell^p, \|\cdot,\cdot\|_p\big)$ is 2-Banach space.

3. Conclusion

Based on discussion above, conclusion for this paper are:
1. $\|\cdot,\cdot\|_p$ is norm-2 that can be defines in ℓ^p vector space.
2. $\big(\ell^p, \|\cdot,\cdot\|_p\big)$ is 2-normed space.

3. $(\ell^p, \|\cdot,\cdot\|_p)$ is 2-Banach space.

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