A Unified Uncertainty Mathematical Model for Input Oriented Data Envelopment Analysis

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Abstract

In a competitive environment, it is crucial for organizations to know how efficiently and effectively they are operating compared to similar organizations. The challenge is to somehow draw helpful insights from all these numbers that will lead to improvements in the performance of the organization. Efficiency measurement is one aspect of organizational performance. Data Envelopment Analysis (DEA) technique is considered the most appropriate tool for evaluating the performance of a set of comparable homogenous organizations under some predefined conditions. In real-life problems, values for input and/or output variables include uncertainty, this uncertainty may be randomness or vagueness in nature. The purpose of this study is the using of some theoretical results to develop a unified DEA model to handle different uncertainty types, the developed model allows various natures of variables (vagueness, randomness and deterministic) depending on the nature of uncertainty in the variables. Implementation of the model was presented through some cases to illustrate the model functionality. In addition, the results are compared with three other different DEA models; a Combined fuzzy/deterministic model, a Combined stochastic/deterministic model, and a deterministic model. Managers can rely on the developed model to assess relative efficiency in business complex systems associated with different uncertainty natures.

Keywords:
Data Envelopment Analysis, Uncertainty Variation, Performance Evaluation, Model Validation, Efficiency Assessment.

1. Introduction

One of the most important principles in any business is the principle of efficiency; where the best possible economic effects (outputs) are attained with as little economic sacrifices as possible (inputs). In order to assess the relative efficiency of a business unit, it is necessary to consider the conditions and operation results of other units of the same kind and to determine the real standing of the results of such a comparison. DEA is a powerful quantitative tool that provides a means to obtain useful information about relative efficiency and performance of firms, organizations, and all sorts of functionally similar, relatively autonomous operating units, known as decision making units (DMUs). The DEA objective is to assess the relative efficiency of each DMU in relation to its peers. The DEA result is a classification of all DMUs as either "efficient" or "inefficient". Not only classifying the DMUs, but also determining the level of inefficiency and the corresponding amount to enhance the performance (El-Demerdash et al., 2013).

Usually the investigated DMUs are characterized by a vector of multiple inputs and multiple outputs. A main advantage of DEA is that it does not require any prior assumptions on the underlying functional relationships between the input and output variables (Cooper et al., 2006). However, one of the most weaknesses in traditional DEA models is that it does not allow uncertainty variations in input and output data, although many important reality observations either stochastic or fuzzy in nature. As a result, DEA efficiency measurement may be sensitive to such variations Cooper et al. (2011). A DMU which is rated as efficient relative to other DMUs may turn inefficient if such uncertainty variations are considered, or vice versa. In another word, if the collected data for a

variable are not represented in the correct form nature, then the resulting efficiencies will be erroneous and misleading because of a high sensitivity of the efficiency scores to the realized levels of inputs or outputs.
The rest of the paper is organized as follows. The coming section includes the literature survey. This is followed by proposed a unified DEA model that handles deterministic variables and variables with different uncertainty variations. Then a hypothetical illustrative example. Then applications of DEA. The paper will end with the conclusions.

2. Literature Review

There are good efforts that have been made recently in DEA model to handle the randomness or vagueness in data. For handling stochastic data, the proposed models extended output-oriented DEA model such that the case of all outputs have a random nature using chance constrained programming while all inputs are deterministic (Land et al., 1993; Desai et al., 2005; Wu et al., 2012) or by taking into consideration the possibility of all variables in random variation (Olesen and Petersen, 1995; Cooper et al., 2004; Razavyan and Tohidi, 2008; Khodabakhshi, 2010; Azadeh et al., 2015; Liu et al., 2017). There are very few researches considered input-oriented DEA model by assuming inputs are deterministic and outputs are random in nature (Talluri et al., 2006) or assume that all input variables are random in nature and all outputs are deterministic (El-Khodary et al., 2010; El-Demerdash et al., 2013; El-Demerdash et al., 2014). it is evident that the developed stochastic DEA models adopted either original or output-oriented DEA model, but still very few considered input-oriented DEA model. It is noticed that, available stochastic DEA models consider either all output variables or all input variables or all variables as random in nature, although some might have a deterministic in nature. In addition to, they were considered there is not mutually correlated relation between same inputs or same outputs.

In the recent years, there are good efforts that have been made in the DEA models to handle the vagueness in variables either fuzzy input or fuzzy output. A fuzzy DEA model was developed either for traditional model (Saati et al., 2002; Liu et al., 2007; Chiang and Che, 2010; Khoshfetrat and Daneshvar, 2011) or extended output-oriented DEA model such that consider all data as fuzzy in nature (Girod, 1996; Kao and Liu, 2000; Entani et al., 2002; Zerafat et al., 2010; Azadeh et al., 2011; Zerafat et al., 2012; Egilmez et al., 2016; and Hatami-Marbini et al., 2017). From surveying the literature, we reached two main conclusions. The first is that the available fuzzy DEA models consider all output and/or input variables as fuzzy in nature, although some might have a deterministic in nature. The second is that the DEA models adopted are either the traditional DEA model or the output-oriented DEA model, but none (to our knowledge) considered input-oriented DEA models. Accordingly, in this study, we attempted to develop a Fuzzy input-oriented DEA that considers a mix of fuzzy and deterministic input and/or output variables.

There is one trial to develop a flexible DEA model deals with different types of variables by A. Azadeh and S.M. Alem (2010) identified the need to combine the deterministic DEA, stochastic DEA, and fuzzy DEA models to establish the FCCDEA model; that can be more useful in real life problems. They presented a flexible DEA – fuzzy DEA (FDEA) – chance constrained DEA (CCDEA) approach. This approach used DEA model when all data are crisp and used FDEA or CCDEA when inputs variables are not crisp and outputs variables are crisp. After collected data, it is checked for the status of data either crisp or non-crisp. FDEA is used for non-crisp data. However, for crisp data the probability level (β) is checked, if it is not equal to 1 then CCDEA is used. Otherwise, used DEA. Based on the proposed approach they deal with three conditions: data is non-crisp, then, use FDEA method, data is crisp, and $\beta \neq 1$, then use CCDEA method, and data is crisp, and $\beta = 1$, then use DEA method. In their work, they tried to combine models, however, they didn't allow the same model to have different types of uncertainty.

This paper forms one part of a series of continuing research efforts. In previous work we treated the topic of stochastic characterizations of efficiency and inefficiency in DEA using chance constrained method to deal with stochastic variation in constraints to convert them to equivalent deterministic nonlinear model. However, we used α-cut approach to deal with fuzzy variation in constraints to convert them to equivalent crisp model. El-Khodary, et al. (2010) developed an algorithm to help any comparable organizations for evaluating their performance, the developed algorithm based on the DEA model and working in a stochastic environment under assumptions that all input variables are random and the relation between each DMU for the stochastic input variables are independent. El-Demerdash, et al., (2013) developed an algorithm for a stochastic chance constrained input-oriented model, where the stochastic inputs are normally distributed, while the remaining inputs and all outputs are deterministic and the relation between the same stochastic input variable through different DMUs is dependent. El-Demerdash et al., (2016) developed a stochastic DEA model that allows some input and/output variables to be stochastic while keeping other variables deterministic. In this model we allowed some dependency between values of the same stochastic variable across different DMUs. Tharwat et al., (2019) developed a fuzzy DEA model that considers a mix of both fuzzy and deterministic output and/or input variables to be solved using the α-cut approach. The developed model algorithm is divided into three stages; it starts by defining the membership function for the fuzzy variables (assumed triangular), then finding the α-cuts for the fuzzy variables, and finally calculating the relative efficiency for each DMU.

The paper in hand is an extension to the previous work, where we need to develop a unified DEA model that consider some of input and output variables have uncertainty nature either stochastic and/or fuzzy and the remaining input and output variables are deterministic.

3. The Developed DEA Unified Model with Uncertainty

In real world problems, exact data may not always be available due to the existence of uncertainty and we needed to evaluate the performance of any comparable institutions to assure the quality given that some of the input and /or output variables might have as uncertainty in nature. So, it was necessary to develop a model to deal with this case. Therefore, in this section, we aim to develop a unified input-oriented DEA model to handle different uncertainty types of variables. We decided to combine previous work in El-Demerdash et al., (2016) and Tharwat et al., (2019). The proposed model allows some input and/output variables to be uncertainty (stochastic/fuzzy) in nature while keeping other variables deterministic. In this section we had concern about input-oriented DEA model which defines the frontier by seeking the maximum possible proportional reduction in input usage, with output levels held constant, for each DMU. the input oriented VRS DEA model for measuring the efficiency level of *DMUp* is as follow:

$$
\begin{aligned}
& Min \quad Z_p = \theta \\
s.t.\quad & \\
& \sum_{i=1}^{n} \lambda_i x_i \leq \theta x_p \quad , \forall j = 1 \ldots m \\
& \sum_{i=1}^{n} \lambda_i y_i \geq y_p \quad , \forall k = 1 \ldots \mathrm{s} \\
& \sum_{i=1}^{n} \lambda_i = 1 \\
& \lambda_i \geq 0, (i = 1,2, \ldots, n)
\end{aligned}
\qquad (M-1)
$$

where k = 1 to 's' (no. of outputs); j = 1 to 'm' (no. of inputs); i = 1 to 'n' (no. of DMUs); y_{ik} = amount of output k produced by DMU i; x_{ij} = amount of input j utilized by DMU i; λ_i = weight given to DMU i.

3.1 Theoretical results

Proposition 1: Assume that some of the input and/or output observations are random variables, the equivalent chance-constraint unified input-oriented DEA model for measuring the efficiency level of p^{th} DMU for the model *(M - 1)* is presented as below:

$$
\begin{aligned}
& Min\, Z_p = \theta \\
s.t.\quad & \\
& pr\left\{\sum_{i=1}^{n} \lambda_i x_{ij} \leq \theta x_{pj}\right\} \geq (1-\alpha_j) \quad , \forall j \in J_S \\
& \sum_{i=1}^{n} \lambda_i x_{ij} \leq \theta x_{pj} \quad , \forall j \in J_D \\
& pr\left\{\sum_{i=1}^{n} \lambda_i y_{ik} \geq y_{pk}\right\} \geq (1-\alpha_k) \quad , \forall k \in K_S \\
& \sum_{i=1}^{n} \lambda_i y_{ik} \geq y_{pk} \quad , \forall k \in K_D \\
& \sum_{i=1}^{n} \lambda_i = 1 \\
& \lambda_i \geq 0, (i = 1,2, \ldots., n)
\end{aligned}
\qquad (M-2)
$$

Where α_j: significance level for input j, α_k: significance level for output k, J_D is the set of deterministic inputs,

J_S is the set of stochastic inputs, J is the set of all inputs, where $J_D \cup J_s = J$ and K_D is the set of deterministic outputs, K_s is the set of stochastic outputs, and K set of all outputs, where $K_D \cup K_s = K$.
Proof: see Land et al. (1993). ■

Proposition 2: Assume that the random input variable ($x_{ij} \in J_s$) are identically normal distributed and pairwise dependant ($cov(x_{ij}, x_{pj}) \neq 0$), then the equivalent deterministic nonlinear model for the unified input oriented DEA model presented in the model (*M-3*) is as:

$$Min\ Z_p = \theta$$

$$s.t.$$

$$\sum_{i=1}^{n} \lambda_i \mu_{ij} - \theta \mu_{pj} \leq e_j \sqrt{(\lambda_p - \theta)^2 \sigma_{pj}^2 + \sum_{\substack{i=1 \\ i \neq p}}^{n} \lambda_i^2 \sigma_{ij}^2 + 2\lambda_i \lambda_p cov(x_{ij}, x_{pj})} \quad , \forall j \in J_S$$

$$\sum_{i=1}^{n} \lambda_i x_{ij} \leq \theta x_{pj} \quad , \forall j \in J_D \qquad (M-3)$$

$$pr\left\{\sum_{i=1}^{n} \lambda_i y_{ik} \geq y_{pk}\right\} \geq (1-\alpha_k) \quad , \forall k \in K_S$$

$$\sum_{i=1}^{n} \lambda_i y_{ik} \geq y_{pk} \quad , \forall k \in K_D$$

$$\sum_{i=1}^{n} \lambda_i = 1$$

$$\lambda_i \geq 0, (i = 1,2, \dots, n).$$

Where μ_{ij} = mean of input j for DMU i, σ_{ij}^2 = variance of input j for DMU i.
Proof: see El-Khodary, et al. (2010). ■

Proposition 3: Assume that the random output variable ($y_{ik} \in K_s$) are identically normal distributed and pairwise dependent ($cov(y_{ik}, y_{pk}) \neq 0$), then the equivalent deterministic nonlinear model for the unified input oriented VRS DEA model presented in the model (*M-4*) is as:

$$Min\ Z_p = \theta$$

$$s.t.$$

$$\sum_{i=1}^{n} \lambda_i \mu_{ij} - \theta \mu_{pj} \leq e_j \sqrt{(\lambda_p - \theta)^2 \sigma_{pj}^2 + \sum_{\substack{i=1 \\ i \neq p}}^{n} \lambda_i^2 \sigma_{ij}^2 + 2\lambda_i \lambda_p cov(x_{ij}, x_{pj})} \quad , \forall j \in J_S$$

$$\sum_{i=1}^{n} \lambda_i x_{ij} \leq \theta x_{pj} \quad , \forall j \in J_D \qquad (M-4)$$

$$\sum_{i=1}^{n} \lambda_i \mu_{ik} - \mu_{pk} \geq e_k \sqrt{\lambda_i^2 \sigma_{pk}^2 + \sum_{\substack{i=1 \\ i \neq p}}^{n} \lambda_i^2 \sigma_{ik}^2 + 2\lambda_i \lambda_p cov(y_{ik}, y_{pk})} \quad , \forall k \in K_s$$

$$\sum_{i=1}^{n} \lambda_i y_{ik} \geq y_{pk} \quad , \forall k \in K_D$$

$$\sum_{i=1}^{n} \lambda_i = 1$$

$$\lambda_i \geq 0, (i = 1,2, \dots, n).$$

Where μ_{ik} = mean of output k for DMU i, σ_{ik}^2 = variance of output k for DMU i.
Proof: El-Demerdash et al., (2016).■

Proposition 4: Assume that some of the input and/or output observations are fuzzy variables, the equivalent fuzzy input-oriented VRS DEA model for measuring the efficiency level of *DMUp* for the model *(M - 1)* is presented as below:

$$Min\ Z_p = \theta$$

$s.t.$

$$\sum_{i=1}^{n} \lambda_i \tilde{x}_{ij} \leq \theta \tilde{x}_{pj} \quad ,\forall j \in J_F$$
$$\sum_{i=1}^{n} \lambda_i x_{ij} \leq \theta x_{pj} \quad ,\forall j \in J_D \qquad (M-5)$$
$$\sum_{i=1}^{n} \lambda_i \tilde{y}_{ik} \geq \tilde{y}_{pk} \quad ,\forall k \in K_F$$
$$\sum_{i=1}^{n} \lambda_i y_{ik} \geq y_{pk} \quad ,\forall k \in K_D$$
$$\sum_{i=1}^{n} \lambda_i = 1$$
$$\lambda_i \geq 0, (i = 1,2, \ldots, n).$$

Where $\tilde{x}_{ij}$: fuzzy number for input j utilized by DMU i, $\tilde{y}_{ik}$: fuzzy number for output k produced by DMU i, J_D is the set of deterministic inputs, J_F is the set of fuzzy inputs, J is the set of all inputs, where $J_D \cup J_F = J$ and K_D is the set of deterministic outputs, K_F is the set of fuzzy outputs, and K set of all outputs, where $K_D \cup K_F = K$.
Proof: see Girod (1996) ■

Proposition 5: Suppose that the fuzzy input variables ($\tilde{x}_{ij} \in J_F$) are follow triangular membership function, then the equivalent crisp model for the input-oriented VRS DEA model using α-cut approach presented in the model (*M-5*) is as:

$$Min\, Z_p = \theta$$

$s.t.$

$$\sum_{i=1}^{n} \lambda_i \tilde{x}_{ij} \leq \theta \tilde{x}_{pj} \quad ,\forall j \in J_F$$
$$\alpha x_{ij}^{M} + (1-\alpha) x_{ij}^{L} \leq \tilde{x}_{ij} \leq \alpha x_{ij}^{M} + (1-\alpha) x_{ij}^{U} \quad ,\forall j \in J_F, i = 1,2, \ldots, n$$
$$\sum_{i=1}^{n} \lambda_i x_{ij} \leq \theta x_{pj} \quad ,\forall j \in J_D \qquad (M-6)$$
$$\sum_{i=1}^{n} \lambda_i \tilde{y}_{ik} \geq \tilde{y}_{pk} \quad ,\forall k \in K_F$$
$$\sum_{i=1}^{n} \lambda_i y_{ik} \geq y_{pk} \quad ,\forall k \in K_D$$
$$\sum_{i=1}^{n} \lambda_i = 1$$
$$\lambda_i \geq 0, (i = 1,2, \ldots, n).$$

Where α: α-cut level for fuzzy variables, x_{ij}^{L}: the lower value of input fuzzy variable j utilized by DMU i, x_{ij}^{M}: median value of input fuzzy variable j utilized by DMU i, x_{ij}^{U}: the upper value of input fuzzy variable j utilized by DMU i.
Proof: Tharwat et al., (2019) ■

Proposition 6: Suppose that the fuzzy output variables ($\tilde{y}_{ik} \in K_F$) are follow triangular membership function, then the equivalent crisp model for the input-oriented VRS DEA model presented in the model (*M-6*) is as:

$$Min\, Z_p = \theta$$

$s.t.$

$$\sum_{i=1}^{n} \lambda_i \tilde{x}_{ij} \leq \theta \tilde{x}_{pj} \quad ,\forall j \in J_F$$
$$\alpha x_{ij}^{M} + (1-\alpha) x_{ij}^{L} \leq \tilde{x}_{ij} \leq \alpha x_{ij}^{M} + (1-\alpha) x_{ij}^{U} \quad ,\forall j \in J_F, i = 1,2, \ldots, n$$
$$\sum_{i=1}^{n} \lambda_i x_{ij} \leq \theta x_{pj} \quad ,\forall j \in J_D \qquad (M-7)$$

$$\sum_{i=1}^{n} \lambda_i \tilde{y}_{ik} \geq \tilde{y}_{pk} \quad ,\forall k \in K_F$$
$$\alpha y_{ik}^{M} + (1-\alpha) y_{ik}^{L} \leq \tilde{y}_{ik} \leq \alpha y_{ik}^{M} + (1-\alpha) y_{ik}^{U} \quad ,\forall k \in K_F, i = 1,2,\dots,n$$
$$\sum_{i=1}^{n} \lambda_i y_{ik} \geq y_{pk} \quad ,\forall k \in K_D$$
$$\sum_{i=1}^{n} \lambda_i = 1$$
$$\lambda_i \geq 0, (i = 1,2,\dots.,n).$$

Where y_{ik}^{L}: the lower value of fuzzy output variable k produced by DMU i, y_{ik}^{M}: median value of fuzzy output variable k produced by DMU i, y_{ik}^{U}: the upper value of the fuzzy output variable k produced by DMU i.
Proof: Tharwat et al., (2019) ■

Proposition 7: Consider an input oriented VRS DEA model that deals with some of variables have different types of uncertainty (i.e. randomness and vagueness) in nature and the recent variables are deterministic in nature, then the equivalent deterministic unified input oriented VRS DEA model presented below:

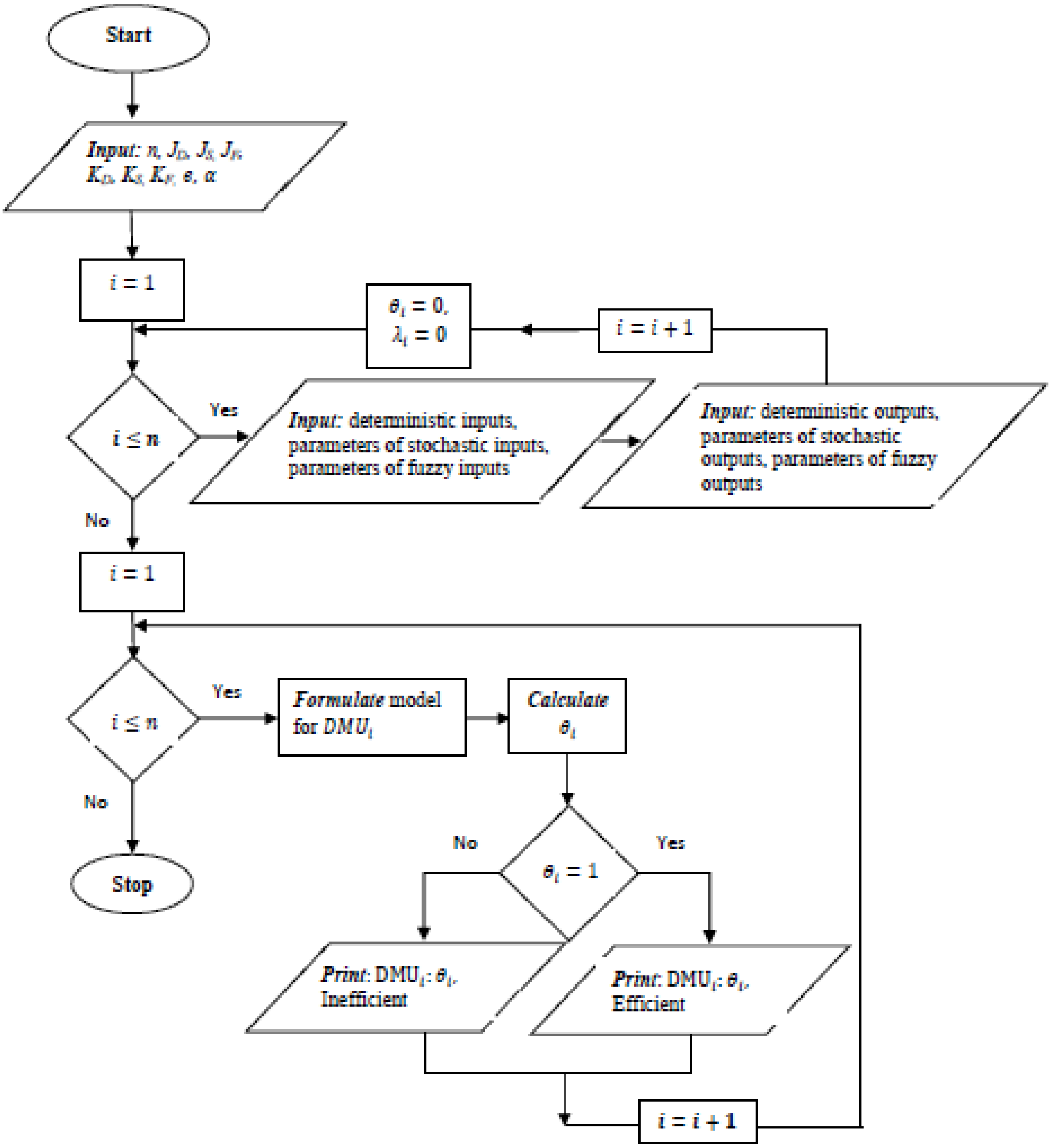
$$Min\ Z_p = \theta$$
$$s.t.$$
$$\sum_{i=1}^{n} \lambda_i \tilde{x}_{ij} \leq \theta \tilde{x}_{pj} \quad ,\forall j \in J_F$$
$$\alpha x_{ij}^{M} + (1-\alpha) x_{ij}^{L} \leq \tilde{x}_{ij} \leq \alpha x_{ij}^{M} + (1-\alpha) x_{ij}^{U} \quad ,\forall j \in J_F, i = 1,2,\dots,n$$
$$\sum_{i=1}^{n} \lambda_i \mu_{ij} - \theta \mu_{pj} \leq e_j \sqrt{(\lambda_p - \theta)^2 \sigma_{pj}^2 + \sum_{\substack{i=1 \\ i \neq p}}^{n} \lambda_i^2 \sigma_{ij}^2 + 2\lambda_i \lambda_p cov(x_{ij}, x_{pj})} \quad ,\forall j \in J_S$$
$$\sum_{i=1}^{n} \lambda_i x_{ij} \leq \theta x_{pj} \quad ,\forall j \in J_D \qquad (M-8)$$
$$\sum_{i=1}^{n} \lambda_i \tilde{y}_{ik} \geq \tilde{y}_{pk} \quad ,\forall k \in K_F$$
$$\alpha y_{ik}^{M} + (1-\alpha) y_{ik}^{L} \leq \tilde{y}_{ik} \leq \alpha y_{ik}^{M} + (1-\alpha) y_{ik}^{U} \quad ,\forall k \in K_F, i = 1,2,\dots,n$$
$$\sum_{i=1}^{n} \lambda_i \mu_{ik} - \mu_{pk} \geq e_k \sqrt{\lambda_i^2 \sigma_{pk}^2 + \sum_{\substack{i=1 \\ i \neq p}}^{n} \lambda_i^2 \sigma_{ik}^2 + 2\lambda_i \lambda_p cov(y_{ik}, y_{pk})} \quad ,\forall k \in K_s$$
$$\sum_{i=1}^{n} \lambda_i y_{ik} \geq y_{pk} \quad ,\forall k \in K_D$$
$$\sum_{i=1}^{n} \lambda_i = 1$$
$$\lambda_i \geq 0, (i = 1,2,\dots.,n).$$

Where $J_D \cup J_S \cup J_F = J$ and $K_D \cup K_S \cup K_F = K$.

Proof: For proposing a unified input oriented VRS DEA model that allow input/output variables to be defined as deterministic, randomness and/or vagueness depending on the nature of uncertainty in the variables. The first constraint in model (M-1) is responsible for deterministic input variables in nature, in the model (M-4) represent the equivalent nonlinear deterministic for the constraint after applied chance constraint approach to be responsible for stochastic input variables in nature. Moreover, in the model (M-7) represent the equivalent linear crisp for the constraint after applied α-cut approach to be responsible for fuzzy input variables in nature.
The second constraint in model (M-1) is responsible for deterministic output variables in nature, in the model (M-4) represent the equivalent nonlinear deterministic for the constraint after applied chance constraint approach to be responsible for stochastic output variables in nature. Moreover, in the model (M-7) represent the equivalent linear crisp for the constraint after applied α-cut approach to be responsible for fuzzy output variables in nature. ■

3.2 The Flow Chart for Unified Input Oriented Data Envelopment Analysis

Figure 1. the flow chart for unified input-oriented DEA



4. Illustrative Example

To illustrate the significance of our unified input-oriented DEA model let's consider this example. We have seven DMUs with three input variables, and two output variables. Two input variables are deterministic (Input 1, Input 2) and one fuzzy (Input 3). The input fuzzy variables are assumed triangular fuzzy numbers having minimum, average and maximum values for each DMU. One output variable is deterministic (Output 1) and one stochastic (Output2). Data considered variable are shown in Table 1, to Table 3 respectively. Furthermore, we assume the α-cut level for the fuzzy variables is 0.5 and the level of significance for stochastic variable is 5% and hence $\boldsymbol{e}$ will be 1.96.

Table 1 Hypothetical data for the deterministic variables for the DMUs

DMU	**Inputs**		**Output**
	Input 1	**Input 2**	**Output 1**
A	6.11	4.36	0.21
B	3.66	2.54	0.12
C	1.44	0.48	0.14
D	1.21	0.23	0.10
E	2.75	1.40	0.10
F	4.18	2.74	0.06
G	6.39	3.36	0.18

Table 2 Hypothetical data for the fuzzy variables for the DMUs

DMU	**Input 3**		
	Lower	**Middle**	**Upper**
A	1.76	7.27	12.27
B	3.85	4.65	5.53
C	1.33	1.88	3.38
D	0.78	1.48	2.06
E	3.22	3.63	4.61
F	4.30	6.13	8.03
G	4.40	8.00	10.68

Table 3. Hypothetical data for the stochastic variable for the DMUs

DMU	**Output 2**				
	$\boldsymbol{\mu}$	$\boldsymbol{\sigma^2}$	***Covariances***		
A	0.19	0.08	$cov(y_A, y_B)$ =0.006	$cov(y_B, y_D) = 0.006$	$cov(y_C, y_G)$ =0.004
B	0.11	0.12	$cov(y_A, y_C) = 0.003$	$cov(y_B, y_E) = 0.006$	$cov(y_D, y_E) = 0.004$
C	0.10	0.06	$cov(y_A, y_D) = 0.004$	$cov(y_B, y_F) = 0.006$	$cov(y_D, y_F) = 0.004$
D	0.07	0.08	$cov(y_A, y_E) = 0.004$	$cov(y_B, y_G) = 0.01$	$cov(y_D, y_G) = 0.006$
E	0.09	0.08	$cov(y_A, y_F) = 0.004$	$cov(y_C, y_D)$ =0.003	$cov(y_E, y_F) = 0.004$
F	0.07	0.08	$cov(y_A, y_G)$ =0.006	$cov(y_C, y_E)$ =0.003	$cov(y_E, y_G) = 0.006$
G	0.18	0.12	$cov(y_B, y_C) = 0.004$	$cov(y_C, y_F)$ =0.003	$cov(y_F, y_G) = 0.006$

The aim of this problem is to determine the relative efficiency of the DMUs according to our developed model. The models are constructed and solved using GAMS (General Algebraic Modeling System) programming language software. The relative efficiency levels for DMUs are as shown in Table 4.

Table 4 Relative efficiency level for each DMU

DMU	**Unified Input Oriented DEA Model**	**DMU**	**Unified Input Oriented DEA Model**
A	0.81	E	1
B	0.87	F	0.66
C	1	G	0.59
D	1		

5. Results Validation

DEA has several strengths (Manzoni and Islam 2007), but the most important is; it can include multiple inputs and outputs variables, there is no functional form relating inputs to outputs. Nevertheless, DEA has its weaknesses. One of the most important of these is that, since DEA is a nonparametric technique, statistical hypothesis tests are difficult to conduct, and assessing the strength or fit of the resulting model is correspondingly difficult. However, the devolved model must be validated. The most approach the researchers have used for validating DEA model is compared with dissimilar DEA models (Azadeh et al., 2015; Wang & Chin, 2011).

For the above illustrative example, we will compare the relative efficiency of our developed model against 3 other different DEA models:

1- Combined fuzzy/deterministic DEA model developed by Tharwat et al., (2019): in which (input 1, input 2 and output 1 are assumed deterministic and (input 3 and output 2) are assumed fuzzy. Mathematically, for output 2 – we took mean values as middle values and assumed values of lower and upper values as $\mu \pm \sigma$.
2- Combined stochastic/deterministic DEA model developed by El-Demerdash et al., (2016): in which (input 1, input 2 and output 1 are assumed deterministic and (input 3 and output 2) are assumed stochastic. Mathematically, for input 3 – we took average of the three values of fuzzy number as mean value and assumed the value of variance and covariance between DMUs.
3- Deterministic DEA model (original model): in which all variable including (input 3, output 2) are assumed constant variables. Mathematically, for input 3 – we took the average values as deterministic values and for output 2 – we took the mean values as deterministic values.

The models are constructed and solved using GAMS) programming language software. The relative efficiency levels for each DMU in four different DEA models are as shown in Table 5.

Table 5 Relative efficiency level for each DMU for Different Models

DMU	Combined fuzzy/deterministic DEA Model	Combined Stochastic/deterministic DEA Model	Deterministic DEA Model
A	1	0.59	1
B	0.36	1	0.40
C	1	1	1
D	1	1	1
E	0.44	1	0.46
F	0.29	0.99	0.29
G	0.80	0.84	1

Examining the results in Table 4 and Table 5, we find that Efficiency results have change significantly across models. There is not trend in differences as for example ***DMU A*** was consider efficient in model 2 and 4 while it is inefficient in our model (0.81) and ***DMU E*** was consider inefficient in some cases and was considered efficient in our model, while it is not similar cases can be found with different models. In conclusion, model assumption regarding the nature of non-deterministic variables significantly affects the resulting efficiency. In another word, regardless the model type, for each inefficient DMU needed to improvement in the relative efficiency levels either decrease in one or more input or increase in one or more output or both. Therefore, it is obvious that the nature of the variable has an influence on the resulting relative efficiency levels.

6. Applications of Data Envelopment Analysis

It is well known that the main purpose of our developed DEA model is to evaluate the relative efficiency of homogeneous DMUs that consider some of input and output variables have uncertainty nature either stochastic and/or fuzzy and the remaining input and output variables are deterministic. The developed DEA model could be used in a great variety of application, for evaluating the efficiencies of many different kinds of entities engaged in many different activities in many different contexts in many different countries such as hospitals, universities, football teams, air force, banks, courts, business firms, and others, including the performance of countries, regions, etc. in addition to, the developed DEA model could be used to determine benchmarks (The set of efficient DMUs was used to establish an internal best practice benchmark to project career development plans for improving the performance of other inefficient DMUs), determine the efficiency for the same company over time, evaluate supply chain network operation efficiency (Improvement in the quality of all supply chain processes lead to cost reductions as well as service enhancement), investigate the efficiency of joining/integrating companies together (through measuring the efficiency of each individual company before merging and expected relative efficiency after the integration), perform before and after study (before to give recommendation for remedial actions, after to measure improvement in efficiency), and provide performance indices or measures for comparing efficiencies, and rate bonds.

7. Conclusions

Since exact data may not always be available in real life problem performance assessments due to the existence of uncertainty. As a result, DEA efficiency measurement may be sensitive to such variations. A DMU which is rated as efficient relative to other DMUs may turn inefficient if such uncertainty variations are considered, or vice versa. In another word, if the collected data for a variable is not represented in the correct form nature, then the resulting efficiencies will be erroneous and misleading because of the high sensitivity of the efficiency scores to the realized levels of inputs or outputs.

Therefore, a unified input-oriented DEA model is proposed to conduct performance assessments in uncertainty environments that able to deal with either input and/or output cases simultaneously, while the variables have vagueness or randomness in nature and the remaining variables are deterministic in nature. The model is tailored for fuzzy variables with triangular membership functions and has been handled using the α-cut approach, while the chance constraints programming approach was used to handle the stochastic variables.

Through the illustrative example, it is noticed that the nature of the variable has an influence on the resulting relative efficiency levels and could toggle the status of the DMU from efficient to inefficient and vice versa. Therefore, it is necessary to identify the nature of the variable from the beginning and apply the appropriate DEA model to achieve reliable results.

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