after 40 -50 years in a 100 years' time scale. In case of birth and treatment the susceptible stage heads to a stable position from where the stage stays constant. The Figure 2 (b) shows the behavior of population infected with HIV only the continuously decrease when it reaches at asymptotical stage. This stage takes less time to stable than susceptible in the HIV only model. The Figure 2 (c) shows the behavior of HIV infected individuals with AIDS symptom. From the figures we see that it increases for first 8-10 years and then decreases until it goes to a stable state. The last two graphs show that when $t \to \infty$ they goes about to zero. A large timescale allows the prevalence of HIV throughout centuries.

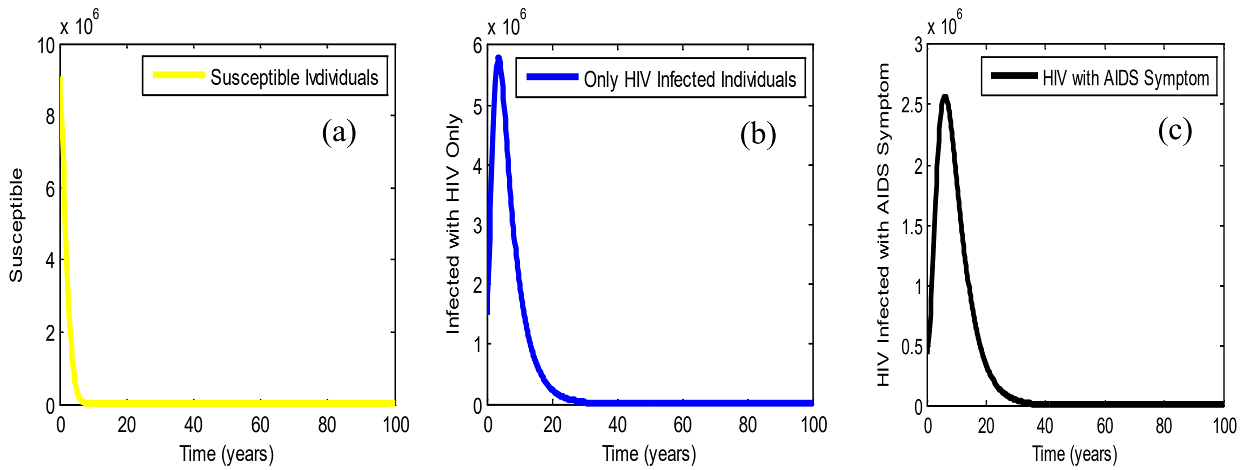



**Figure 3:** The graph for HIV sub model of Susceptible $(S)$, Infected with HIV only $(I_H)$ and HIV infected displaying AIDS symptom $(H_S)$ where time taken as years and $\mu = 0.01, \rho_1 = 0.2, \; d_2 = 0.333, \; \eta_3 = 1.02$ and $\beta_1 = 0.95$.

The Figure 3 (a) shows that if we gradually increase the value of $\beta_1$ susceptible individuals go to stable label faster than the small value of $\beta_1$. The Figure 3 (b) shows a great impact for the HIV infected individuals for the changing value of $\beta_1$. For the small value of the HIV contact rate the HIV infected compartment decreases continuously but for the greater value of $\beta_1$ it increases at first and then starts to decreases swiftly. The stable time scale of the susceptible and the HIV infected individuals are vice versa. Figure 3 (c) shows the same graphical feature for the different values of $\beta_1$ but if we increase the value of $\beta_1$ the HIV infected individuals showing AIDS symptom take less time to increase and then decrease. The time period of reaching stable situation is clearly decreased due to increase of $\beta_1$.

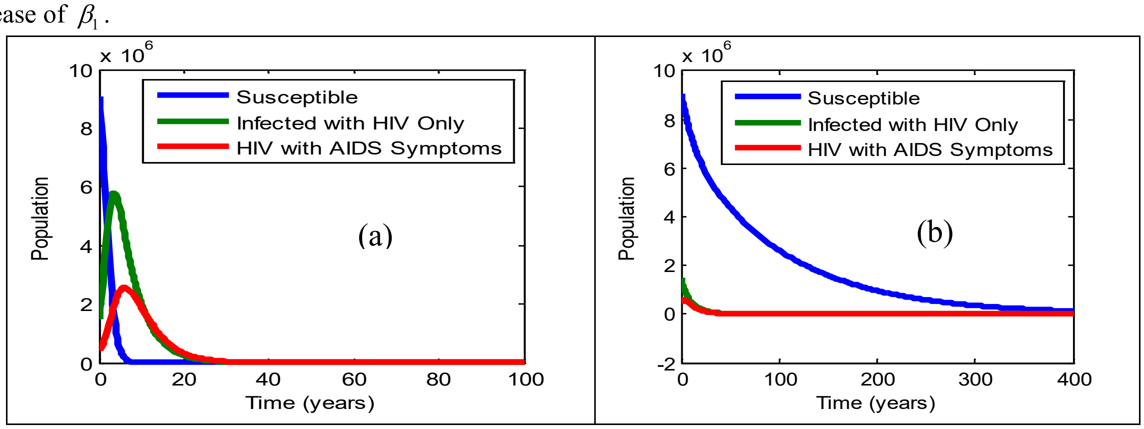



**Figure 4:** Graphs of HIV prevalence where $\beta_1 = 0.95$ and $R > 1$ for the first plot and $\beta_1 = 0.075$, $R < 1$ for the second plot and all other parameters values are same as in Figure 3.

The first graph of Figure 4 represents the stability analysis of the HIV sub model of the endemic equilibrium where $\beta_1 = 0.95$. Since for $t \to \infty$ the state variable converge to equilibrium and it happens numerically for $R_1 > 1$. Hence the endemic equilibrium exists. The second graph of Figure 4 represents the stability analysis of the disease free equilibrium at $\beta_1 = 0.075$ for $R_1 < 1$. Numerically it is shown that when $t \to \infty$ then all other disease except the susceptible die out. So HIV cannot persist if $R_1 < 1$ and for the lower value of $\beta_1$. It shows that the increase of the value of $R_1$ accelerates the prevalence of HIV.

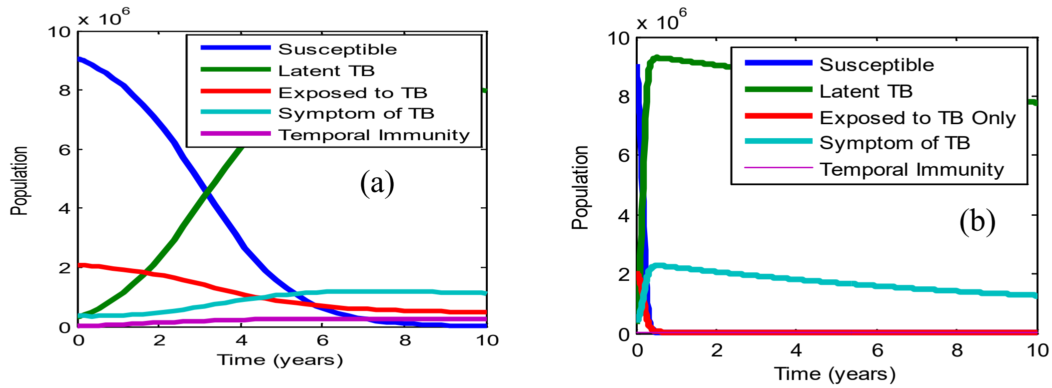
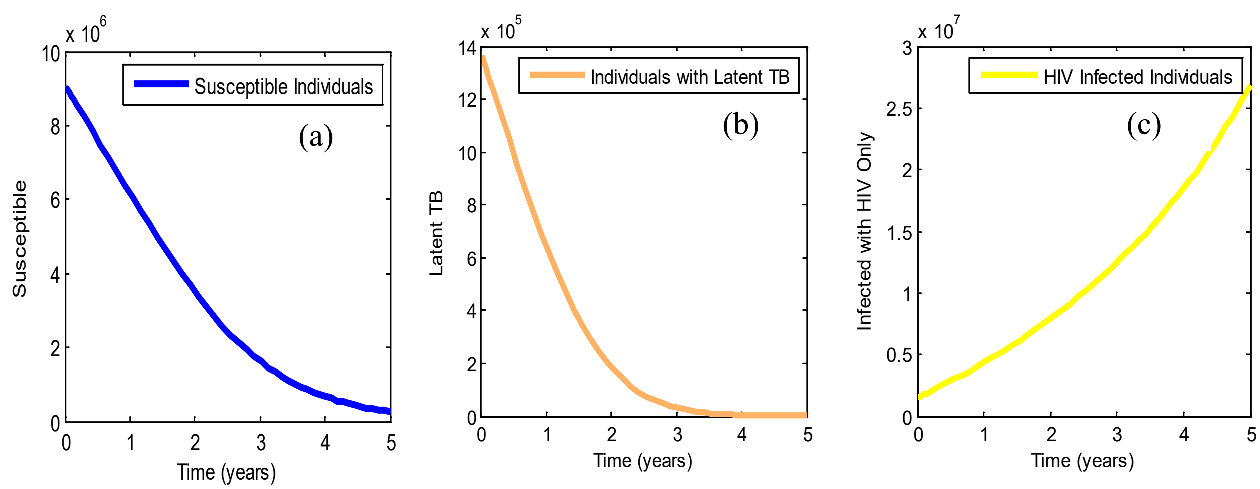



**Figure 5:** The impact of the TB sub model for 10 years where $\beta_2 = 0.6$ and $\beta_2 = 6.0$. All other parameter's values are as same as the Figure 4.

Figure 5 shows the impact and effect of the TB sub model for different values of $\beta_2$ over a 10 years' time period. We see that the decrease of susceptible individuals leads the $E_{T0}$ individuals faster than any other compartment in the TB sub model. From the first graph of Figure 5 it is clear that for the larger value of $\beta_2$ the $E_{T0}$ compartment increase more swiftly than the small value of $\beta_2$. The $S_T$ compartment takes more time to be stable for the lower value of effective contact rate. For the larger value of $\beta_2$ the latent TB class reach sable state within very short time and therefore increase in $E_{T0}$. Figures show the variation of recovery with temporal immunity for the variation of $\beta_2$ if we increase the value of $\beta_2$ then the recovery with temporal immunity decrease and vice versa.



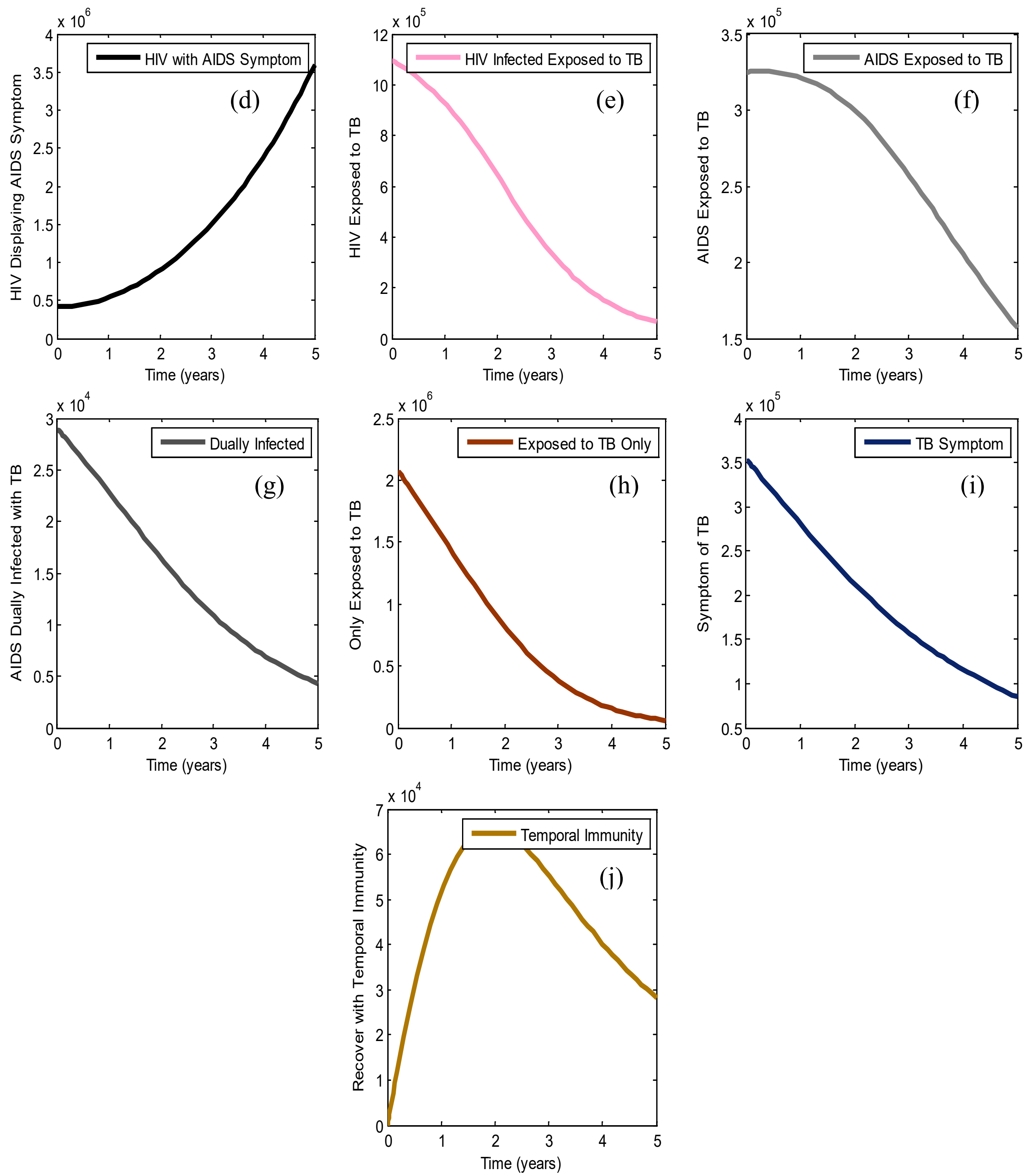



**Figure 6:** Graph of the coinfection model with $\beta_1 = 0.075$ and $\beta_2 = 0.35$. Graph showing (a) the Susceptible $S$, (b) Laten TB $L_T$, (c) Infected with HIV only $I_H$, (d) HIV infected displaying AIDS symptom $H_S$, (e) HIV infected individuals exposed to TB $E_{TH}$, (f) AIDS individuals exposed to TB $E_{HT}$, (g) AIDS individuals dually infected with TB $H_{DT}$, (h) Exposed to TB only $E_{T0}$, (i) Symptom of TB $S_T$ and (j) Recovery with temporal immunity $R_T$.

Figure 6 shows the behavior of infected individuals at various stages of HIV/AIDS and TB co-infection for 5 years. Graphs represent the prevalence of HIV, TB and their co-infection for different values of $\beta_1$ and $\beta_2$. Figure 6(a) shows that the great number of population reduce from susceptible individuals for the prevalence of HIV and TB. Figure 6(b) shows that effect of the reduction of susceptible individuals directly hit upon $L_T$ and leads to the increase of latent TB. Figure 6(c) shows that $I_H$ compartment increase over the time. Figure 6(d) shows the behavior of HIV infected individuals showing AIDS symptom slightly increase but after 1-2 years it shows the symptoms very fast in 5 years time scale. Figure 6(e) shows the HIV infected individuals exposed to TB that means it is a co-infection compartment and it takes 5 or more years to get stable position for our model. Figure 6(f) shows the behavior of AIDS infected individuals exposed to TB that goes dramatically asymptotical low level. Figure 6(g) describes the interaction of the AIDS individuals dually infected with TB. The individuals of this stage might be the combination of AIDS and TB in which the number of individuals dually sick from AIDS and TB to asymptotically low levels. Figure 6(h) shows the behavior of the people who are exposed to TB only which reaches asymptotically level state faster than any other stages of the model. Figure 6(i) shows the behavior of the individuals with symptom of TB only and reaches asymptotically level where it is more or less constant. Figure 6(j) shows that the people who are recovered from TB but not dually infected with HIV. At first recovered compartment goes to high and then starts to decrease due to the HIV infected individuals.

Table 1: Descriptions of parameters and their values (David 2015, Bhunu et al. 2009, Samad and Biswas 2018)

| **Parameters** | **Descriptions** | **Values** | **Parameters** | **Descriptions** | **Values** |
|---|---|---|---|---|---|
| A | Recruitment rate of Susceptible Individuals | 0.029 | $\eta_1$ | Modification Parameter | 1.2 |
| C | Contact Rate | 3 | $\eta_2$ | Modification Parameter | 1.05 |
| $c_1$ | Number of Contacts for Active TB Class | 100 | $\eta_3$ | Modification Parameter | 1.02 |
| $c_2$ | Number of Contract having only HIV and no TB | 2.00 | $\eta_4$ | Modification Parameter | 1.1 |
| $c_3$ | Number of Contract having HIV and TB | 0.50 | $\delta$ | Modification Parameter | 1.03 |
| $d_1$ | Death Rate Related to TB | 0.1 | $\in$ | Modification Parameter | 1.2 |
| $d_2$ | Death Rate Related to AIDS | 0.333 | $\kappa_1$ | TB Propagation Rate Infected with AIDS | 0.000113 |
| $\beta_1$ | Contact Rate for HIV | 0.074[0.011-0.95] | $\kappa_2$ | Rate of Propagation from HIV to TB Class | 0.0001 |
| $\beta_2$ | Contact Rate of TB | 0.35[0.1-0.6] | $\kappa_3$ | Rate of HIV Positive from TB Progress to AIDS | 0.25 |
| $r_1$ | Natural Recovery Rate | 0.2 | $\kappa_4$ | HIV Infected Rate with Mtb Progress to AIDS | 0.102 |
| $r_2$ | Relapsing Rate for the Individuals with Symptom of TB | 0.00001 | $\kappa_5$ | AIDS Cases Infected Rate with Mtb Progress | 0.0002 |
| $\psi_1$ | Modification Parameter | 1.07 | $\nu_1$ | Rate of Propagation from Laten TB Class to Active Class | 0.009 |
| $\psi_2$ | Modification Parameter | 1.101 | $\mu$ | Natural Mortality Rate | 0.01 |
| $\psi_3$ | Modification Parameter | 0.71 | | | |

**Conclusions**

The ultimate result of this study shows that the dynamic behavior of co-infected people mostly depends on force of infection. In this study we use a nonlinear system whose validity is proved by positivity test and numerical simulations. We denote the population of susceptible by $S$ , the population of Laten TB by $L_T$ , the population of

HIV only by $I_H$, the population of HIV infected displaying AIDS symptom by $H_S$, the population of HIV infected individuals exposed to TB by $E_{TH}$, the population of AIDS individuals exposed to TB by $E_{HT}$, the population of AIDS individuals dually infected with TB by $H_{DT}$, the population of Exposed to TB only by $E_{T0}$, the population of Symptom of TB by $S_T$ and the population of Recovery with temporal immunity by $R_T$. TB can be associated at any stages of HIV/AIDS. The HIV only model is locally asymptotically stable disease free equilibrium when $R_1 < 1$ and locally asymptotically stable endemic equilibrium when $R_1 > 1$. The TB only model is locally asymptotically stable disease free equilibrium when $R_2 < 1$ and locally asymptotically stable endemic equilibrium when $R_2 > 1$. The full model is globally asymptotically stable disease free equilibrium when $R_0 < 1$ and locally asymptotically stable endemic equilibrium when $R_0 > 1$. In this study we use $\beta_1$ and $\beta_2$ as variable and we show in the Figure ) that the change of these variable can affect the basic reproduction number. Numerical simulations show that the HIV only model becomes more dynamic for the increasing number of $\beta_1$. The basic reproduction number of the full model that is assumed in the analytical calculation is clearly $R_1$. The Figures 2 and 3 suggest that the scenario of HIV only model accelerates for the larger value of $\beta_1$. The data that we use in the qualitative analysis is not for a particular community and it declares the numerical simulation purely hypothetic. From this study we also conclude that if the individuals have lower infection rate the better scenarios are in a community for HIV /AIDS, TB and HIV-TB co-infection.

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