

Proposition 3: The upper bound of the bullwhip effect is a function of α and equals $(1+\alpha)/(1-\alpha)$.

Proof: The proof of Proposition 3 is provided in Appendix 2. ■

Remark. From the above proposition it can be seen that an upper bound exists for the bullwhip effect measure of the INAR(1) demand. This has also been plotted in Figure 3.

From (4) we can determine the autocorrelation function (ACF) of the demand process; $ACF = \alpha^k$. This is exactly the same form of the ACF for the AR(1) process. Therefore, the auto-regressive parameter α controls the form of the demand series and its auto-correlation. Moreover, the auto-regressive parameter, α , also controls the presence of zero observations in a demand series. Lower values of α and λ generate a time series with high frequency of zero demands but the bullwhip effect will always exist regardless the presence of zero demand values.

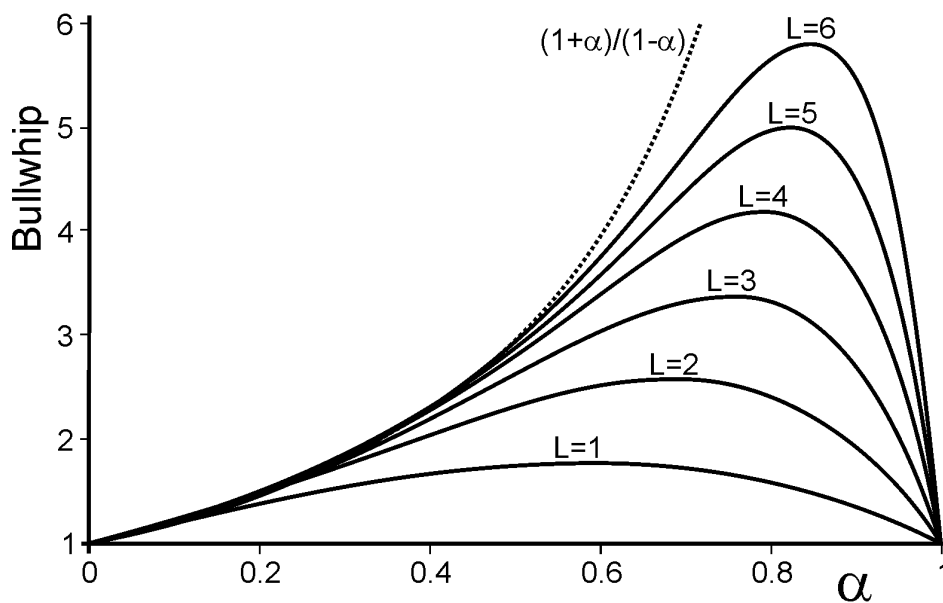


Figure 3. The Bullwhip effect in the OUT policy under INAR(1) demand

4. Conclusions

In this research, we have analytically examined the behavior of the bullwhip measure for an integer-valued INAR(1) demand series in a simple supply chain consisting of an OUT policy with MMSE forecasts. The bullwhip effect measure is first derived and then the conditions under which bullwhip effect exists are discussed. The impact of the auto-regressive demand parameter and lead time are analyzed. The main findings can be summarized as follows:

1. The bullwhip effect exists regardless of the autoregressive parameter and the lead time. There exists a lower bound of $B = 1$ obtained at $\alpha = 0$ and $\alpha = 1$. The amount of bullwhip generated depends on autoregressive and lead time values. For a given lead time L , and $0 < \alpha < 1$, the bullwhip measure first increases with α , it then reaches a maximum value and decreases towards unity at $\alpha = 1$.
2. There exists an upper bound for the bullwhip which is a function of the autoregressive parameter, α . For a given value of α , the upper bound represents the maximum value of the bullwhip effect regardless of the large lead time. The upper bound is tight when α is small.
3. The value of α also controls the intermittency of the demand series. Low values of α (and λ) lead to intermittent demand series that may contain a high proportion of zero values. Results show that the bullwhip effect measure of a highly intermittent series is lower compare to a normal demand series. i.e. the upper bound of the bullwhip effect for $\alpha = 0.05$ equals to $B = 1.1$ while for $\alpha = 0.95$ equals to $B = 39$. However, bullwhip will always exist for intermittent demand.

4. The structure of OUT policies response to the INAR(1) demand and its properties is similar to those of an AR(1) demand. The bullwhip measures take the same form, with $\alpha = \phi > 0$, where ϕ is the auto-regressive parameter in the AR(1) process.

Finally, we note that in our model the forecasts and orders are no longer integer, even though the demand is. While there is no need for the forecasts to be integer (it is internal calculation within the replenishment decision), presumably, if we can only sell integer numbers of products, we can only make integer numbers of products. Thus, there is a need to either a) round the orders to integers in some way or b) to forecast and place orders in integers. For option a) there are at least four options for rounding could be considered: rounding down, rounding up, rounding to the nearest integer, and stochastic rounding. For option b) perhaps by using the thinning operator, or Markov Chains, we can find alternative OUT policy formulations that generate only integer orders. The exploration of these issues are left for further work.

5. References

- Al-Osh, M., Alzaid, A., 1988. Integer-valued moving average (INMA) process. *Statistical Papers* 29, 281–300.
- Box, G., Jenkins, G., 1970. *Time Series Analysis: Forecasting and Control*. San Francisco: Holden-Day.
- Chen, Y.F., Disney, S.M., 2007. The myopic order-up-to policy with a feedback controller. *International Journal of Production Research* 45 (2), 351–368.
- Chen, Y.F., Drezner, Z., Ryan, J.K., Simchi-Levi, D., 2000a. Quantifying the Bullwhip effect in a simple supply chain: The impact of forecasting, lead times, and information. *Management Science* 46 (3), 436–443.
- Chen, Y.F., Ryan, J.K., Simchi-Levi, D., 2000b. The impact of exponential smoothing forecasts on the bullwhip effect. *Naval Research Logistics* 47 (4), 269–286.
- Dejonckheere, J., Disney, S.M., Lambrecht, M., Towill, D.R., 2003. Measuring and avoiding the bullwhip effect: A control theoretic approach. *European Journal of Operational Research* 147 (3), 567–590.
- Disney, S.M., Lambrecht, M.R., 2008. On replenishment rules, forecasting and the bullwhip effect in supply chains. *Foundations and Trends in Technology, Information and Operations Management*, 2 (1), 1–80.
- Duc, T.T.H., Luong, H.T., Kim, Y.-D., 2008. A measure of bullwhip effect in supply chains with a mixed autoregressive-moving average demand process. *European Journal of Operational Research* 187 (1), 243–256.
- Gaalman, G., Disney, S.M., 2006. State space investigation of the bullwhip problem with ARMA(1,1) demand processes. *International Journal of Production Economics*, 104 (2), 327–339.
- Graves, S.C., 1999. A single-item inventory model for a non-stationary demand process. *Manufacturing and Service Operations Management*, 1 (1), 50–61.
- Hosoda, T., Disney, S.M., 2006. The governing dynamics of supply chains: The impact of altruistic behavior. *Automatica* 42 (4), 1301–1309.
- Janjić, A.D., Ristić, M.M., Nastić, A.S., 2014. Mixed thinning INAR(1) model. Lecture notes available for download at <https://web.math.pmf.unizg.hr/cqd/files/Mixed%20thinning.pdf>. Verified 29 May 2017.
- Lee, H.L., So, K.C., Tang, C.S., 2000. The value of information sharing in a two-level supply chain. *Management Science* 46 (5), 626–643.
- Silva, M.E. and Oliveira, V.L., 2004. Difference equations for the higher-order moments and cumulants of the INAR(1) model. *Journal of Time Series Analysis* 25 (3), 317–333.
- Silva, N., Pereira, I., Silva, M.E., 2009. Forecasting in INAR(1) model. *Revstat Statistical Journal* 7 (1), 119–134.
- Wang, X., Disney, S.M., 2016. The bullwhip effect: Progress, trends, and directions. *European Journal of Operational Research* 250 (3), 691–701.

Appendix 1: The auto-covariance function of INAR(1) demand

We can show that the mean of d_t is given by,

$$E(d_t) = E(\alpha \circ d_{t-1} + z_t) = E(\alpha \circ d_{t-1}) + E(z_t) = \alpha E(d_{t-1}) + \lambda. \quad (30)$$

As an INAR(1) process is stationary, $E(d_t) = E(d_{t-1})$, and (30) reduces to

$$E(d_t) = \frac{\lambda}{1-\alpha}. \quad (31)$$

The variance of demand at period t is defined as

$$\begin{aligned}
 \text{Var}(d_t) &= \text{Var}(\alpha \circ d_{t-1} + z_t) \\
 &= \text{Var}(\alpha \circ d_{t-1}) + \text{Var}(z_t) + 2\text{Cov}(d_{t-1}, z_t) \\
 &= \text{Var}(\alpha \circ d_{t-1}) + \lambda \\
 &= E(\alpha \circ d_{t-1})^2 - (E(\alpha \circ d_{t-1}))^2 + \lambda.
 \end{aligned} \tag{32}$$

Using the properties of the thinning operator, (3) we can simplify (32) as follows,

$$\begin{aligned}
 \text{Var}(d_t) &= \alpha^2 E(d_{t-1})^2 + \alpha(1-\alpha)E(d_{t-1}) - \alpha^2 (E(d_{t-1}))^2 + \lambda \\
 &= \alpha^2 (E(d_{t-1})^2 - (E(d_{t-1}))^2) + \alpha\lambda + \lambda = \alpha^2 \text{Var}(d_{t-1}) + \lambda(\alpha + 1).
 \end{aligned} \tag{33}$$

because of stationarity, we have $\text{Var}(d_t) = \text{Var}(d_{t-1})$, yielding,

$$\text{Var}(d_t) - \alpha^2 \text{Var}(d_t) = \lambda(\alpha + 1). \tag{34}$$

Collecting together terms provides

$$\text{Var}(d_t) = \frac{\lambda(\alpha + 1)}{(1 - \alpha^2)} = \frac{\lambda(\alpha + 1)}{(1 - \alpha)(1 + \alpha)} = \frac{\lambda}{(1 - \alpha)}. \tag{35}$$

By recursive substitutions of d_{t-k} for $k \geq 1$, (1) can be written as

$$d_t = \alpha^k \circ d_{t-k} + \sum_{j=0}^{k-1} \alpha^j \circ z_{t-j}. \tag{36}$$

The auto-covariance of lag $k \geq 1$ can then be calculated as

$$\gamma_k = \text{Cov}(d_t, d_{t-k}) = \text{Cov}\left(\alpha^k \circ d_{t-k} + \sum_{j=0}^{k-1} \alpha^j \circ z_{t-j}, d_{t-k}\right) = \text{Cov}(\alpha^k \circ d_{t-k}, d_{t-k}) + \text{Cov}\left(\sum_{j=0}^{k-1} \alpha^j \circ z_{t-j}, d_{t-k}\right). \tag{37}$$

As $\text{Cov}(\alpha \circ d_{t-k}, d_{t-k}) = \alpha \text{Cov}(d_{t-k}, d_{t-k})$,

$$\gamma_k = \alpha^k \text{Cov}(d_{t-k}, d_{t-k}) + \text{Cov}\left(\sum_{j=0}^{k-1} \alpha^j \circ z_{t-j}, d_{t-k}\right). \tag{38}$$

As the correlation between d_{t-k} and z_{t-j} for all $j \leq k-1$ is equal to zero, $\text{Cov}(d_{t-k}, \sum_{j=0}^{k-1} \alpha^j \circ z_{t-j}) = 0$, therefore the auto-covariance function of lag $k \geq 1$ for an INAR(1) process is

$$\gamma_k = \alpha^k \text{Cov}(d_{t-k}, d_{t-k}) = \alpha^k \gamma_0. \tag{39}$$

Appendix 2: Bullwhip effect bounds

Recall, from (29), the bullwhip ratio,

$$B = 1 + \left(2\alpha(1-\alpha^L) + \frac{2(\alpha(1-\alpha^L))^2}{1-\alpha} \right). \quad (40)$$

Given $0 < \alpha < 1$ and $L \geq 1$, we have

$$0 < \alpha^L < 1 \quad (41)$$

Multiplying (40) by minus one (-1) and adding plus one (+1) to both sides, we have

$$0 < 1 - \alpha^L < 1 \quad (42)$$

given $\alpha > 0, 1 - \alpha > 0$ are positive parameters, we get

$$0 < \alpha(1 - \alpha^L) < \alpha. \quad (43)$$

Given (43) and that $\alpha > 0$, we have

$$0 < 2\alpha(1 - \alpha^L) < 2\alpha. \quad (44)$$

Squaring (44) leads to

$$0 < 2(\alpha(1 - \alpha^L))^2 < 2\alpha^2. \quad (45)$$

Knowing that $\frac{1}{1-\alpha} > 1$, we get

$$0 < \frac{2(\alpha(1 - \alpha^L))^2}{1 - \alpha} < \frac{2\alpha^2}{1 - \alpha}. \quad (46)$$

By adding (44) and (46)

$$0 < 2\alpha(1 - \alpha^L) + \frac{2(\alpha(1 - \alpha^L))^2}{1 - \alpha} < 2\alpha + \frac{2\alpha^2}{1 - \alpha}. \quad (47)$$

By simplify (47)

$$0 < 2\alpha(1 - \alpha^L) + \frac{2(\alpha(1 - \alpha^L))^2}{1 - \alpha} < \frac{2\alpha}{1 - \alpha}. \quad (48)$$

To obtain the bullwhip expression, we add plus one (+1) to (48)

$$1 < 1 + \left(2\alpha(1-\alpha^L) + \frac{2(\alpha(1-\alpha^L))^2}{1-\alpha} \right) < 1 + \frac{2\alpha}{1-\alpha}. \quad (49)$$

Therefore the upper and lower bound of the bullwhip effect measure of an INAR(1) is

$$1 < B < \frac{1+\alpha}{1-\alpha}, \quad (50)$$

revealing that B is always greater than one and less than $(1+\alpha)/(1-\alpha)$. It means that bullwhip exists regardless the lead time and the auto-regressive parameter.

Biography

Bahman Rostami-Tabar. Dr. Bahman Rostami-Tabar is a Lecturer in Logistics and Operations Management at Cardiff Business School, Cardiff University. Bahman was previously Lecturer in Supply Chain Management at Coventry University (Aug. 2015-Sep. 2016). Bahman worked as a researcher in LGI lab at Ecole Centrale Paris (Sep. 2013-Aug. 2015) with Faurecia (the world's 6th-largest automotive equipment supplier) where he conducted research on logistics and information sharing. He was a Research Assistant between January 2011 and August 2013 at Kedge Business School, Bordeaux, France.

Bahman holds a Ph.D. in Production Engineering from the University of Bordeaux. He moved to France in 2009 and received an M.Sc. in Information Systems from ECE Paris in 2010. He received the MIM best paper award (IFAC, 2013) and was awarded a Campus France Scholarship. Bahman's research goals are directed towards the use of Operations Research (OR) techniques to improve decision making in supply chains and operations management, and as such positively contribute to both industrial and societal advancements. Bahman's research to date reflects collaboration with more than eight universities around the globe.

Stephen M. Disney. Professor Stephen Disney, Ph.D., is a member of, and former Department Chair of the Logistics and Operations Management Department at Cardiff Business School. He currently leads the Logistics Systems Dynamics Group. He has recently returned from research sabbatical at the University of California, Los Angeles. He has previously held visiting positions at the Chinese University of Hong Kong and Boston University. Professor Disney lectures Operations Management to MBA and MSc students at Cardiff Business School. He has also taught Supply Chain Modelling to the Mathematics Department of Cardiff University. He has extensive experience of teaching in-class, on-line and on-site to both postgraduate and executive audiences.

Professor Disney's current research interests involve the application of control theory and statistical techniques to operations management and supply chain scenarios to investigate their dynamic, stochastic and economic performance. Stephen has a particular interest in the bullwhip effect. He has advised several of the world's largest corporations on this problem. He has worked with many companies in the UK (including Tesco), US (including Lexmark) and Europe (including P&G) and on supply chains that operate globally.