Optimal Planning and Management of Groundwater Level Declination: A Mathematical Model

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Abstract

Groundwater is an essential natural resource of our mother earth that constitutes about 95 per cent of the freshwater on our planet, making it fundamental to human life and economic development.Groundwater level change for many reasons. Some changes are due to natural phenomena, and others are caused by man's activities. It has been declining in Bangladesh since the introduction of deep tube wells (DTWs) and shallow tube wells (STWs) in late 1970s.We formulate a mathematical model to optimize the groundwater level declination with the help of a system of nonlinear ordinary differential equations (ODEs). The model is analyzed by using the stability theory of non-linear differential equations and numerical simulations. We consider two controls, building storage system and diverting river stream instead of frequent pumping of water for the optimal planning and management of groundwater level. We want to find the control strategies so that the overall cost of the storage system and diverting river stream is minimized while maximizing the groundwater level. Finally numerical simulations will be performed to show the effectiveness of the management of groundwater level.

Keywords
Groundwater, Optimal Planning and Management, Storage system, Equilibrium point, Optimal control.

1 Introduction:

Groundwater is one of the most valuable natural resources, which supports human health, economic development and ecological diversity. It has become an immensely important and dependable source of water supplies in all climatic regions including both urban and rural areas of developed and developing countries(Brammer 2009). The contribution from groundwater is vital; perhaps as many as two billion people depend directly upon aquifers for drinking water, and 40 per cent of the world's food is produced by irrigated agriculture that relies largely on groundwater (Hoque 2007). Bangladesh, a small country is blessed with plenty of water resources being located in the basins of mighty Ganges, Meghna, Barhmaputra and Karnaphuli rivers. With numerous rivers, Bangladesh is also affluent in groundwater resource. Since last couple of decades, groundwater is being extensively used for drinking, irrigation and several other purposes eventually declining the ground water level. Groundwater is a vital input for sustaining crop production. Irrigation is the most important water use sector accounting for about 70 percent of the global fresh water withdrawals and 90 percent of consumptive water uses. Groundwater level change for many reasons. Some changes are due to natural phenomena, and others are caused by man's activities. It has been declining in Bangladesh since the introduction of deep tube wells (DTWs) and shallow tube wells (STWs) in late 1970s. Seasonal variation of groundwater has been shifted up to 20 meters on average during last 34 years. During Boro seasons, groundwater table dropped drastically than other seasons. Moreover, excessive demand of water, evaporation, pollution, deforestation, poor storage, low rainfall, urbanization, frequent pumping of water are the important causes of groundwater declination.

Mathematical modeling plays an incredible role for providing quantitative insight into multiple fields. It has already contributed to a better understanding of the mechanisms in various field nowadays (Biswas 2017). Mathematical modeling has gotten attention because modeling and simulation of any physical phenomena allows us for rapid assessment. So it is mainly used to describe the real phenomena which lead to design better prediction, management and control strategies. Mathematical modeling and optimal control are strictly related to each other . Optimal control is the problem of finding a control law for a system so that a certain objective functional is achieved by maximizing or minimizing a particular cost functional (a function of state and control variables) (Lenhart 2007).

In this study, we would like to propose a mathematical model to study the optimal planning and management of groundwater level declination in the south-western region of Bangladesh. We formulate a mathematical model to optimize the groundwater level declination with the help of a system of nonlinear ordinary differential equations (ODEs). We consider two controls for the optimal planning and management of groundwater level. We like to determine the groundwater level in different seasons and study the existence and stability of the model. Finally numerical simulations will be performed to show the effectiveness of the management of groundwater level.

2 Modeling Formulation:

Let us consider $A(t)$, $S(t)$ and $G(t)$ are the state variables that represent atmospheric water, surface water and groundwater.We assume that, water precipitates from atmosphere to surface at the rate α, the infiltration rate from surface to groundwater be β and γ_1, γ_2 are the evaporation rate. The surface water pollutes at the rate ψ . δ and φ be the deforestation rate and rate of frequent pumping of water. The interaction of atmosphere, surface and ground water is shown in the following diagram.

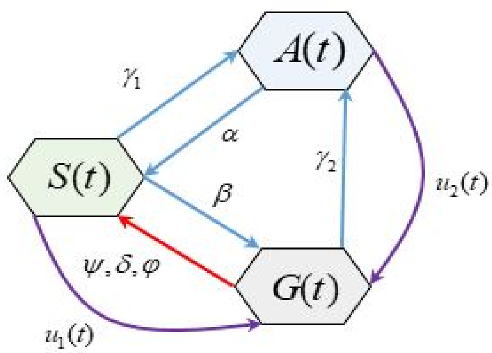

Figure 1: Flow diagram of groundwater level management

Considering the above assumptions and flow diagram, we formulate the following system of nonlinear differential equations:

$$\frac{dA}{dt} = \gamma_1 S(t) + \gamma_2 G(t) - \alpha A(t) \qquad (1)$$

$$\frac{dS}{dt} = \alpha A(t) - \beta G(t)S(t) - \gamma_1 S(t) + \psi G(t)S(t) + \phi G(t) \qquad (2)$$

$$\frac{dG}{dt} = \beta G(t)S(t) - \gamma_2 G(t) + (1-\psi)G(t)S(t) - \varphi G(t) - \delta G(t) \qquad (3)$$

with initial conditions $A(0) = A_0,\ S(0) = S_o\ \ and\ \ G(0) = G_0$ (4)

3 Analytical Solution

3.1 Positivity of the solution:

Lemma-1: If $A_0 \geq 0$, $S_0 \geq 0$ and $G_0 \geq 0$ then the solution $A(t)$, $S(t)$ and $G(t)$ of the system of differentials equations (1)-(3) are positive.

Proof: We have from equation (2) ,

$$\frac{dS}{dt} = \alpha A(t) - \beta G(t)S(t) - \gamma_1 S(t) + \psi G(t)S(t) + \phi G(t)$$

To find the positivity of equations, we have neglected the nonlinear term from (2),

$$\frac{ds}{dt} \geq -\gamma_1 S(t), \Rightarrow \frac{ds}{dt} + \gamma_1 S(t) \geq 0$$

The integrating factor is, I.F $= e^{\int \gamma_1 dt} = e^{\gamma_1 t}$

Multiplying the I.F and closed the required equation,

$S(t) \geq ce^{-\gamma_1 t}$ where, c is integrating constant.when $t = 0,$ then $S(0) = S_0$

Using these values we have $S_0 \geq c$

Thus, we obtain

$S(t) \geq S_0 e^{-\gamma_1 t}$, Since, $S_0 \geq 0$ and γ_1 is very small

Therefore, $S(t) \geq 0$ for all $t \geq 0$.

Similarly, $A(t)$ and $G(t)$ of the system of differentials equations are positive.

3.2 Boundedness of the model:

Lemma 2: The feasible set, $\Omega = \left\{ A(t), S(t), G(t) : w(t) = A(t) + S(t) + G(t), 0 < w(t) < \frac{(\gamma - \delta)}{\eta} \right\}$

where, $\gamma = \min(\gamma_1, \gamma_2)$ and η is a constant, is a region of attraction for the mathematical model and attracts all solutions initiating in the interior of the positive octant.

Proof: Adding the first and second equation of the model (1)-(3), we get

$$\begin{aligned} \dot{A}(t) + \dot{S}(t) &= \gamma_2 G(t) - \beta G(t) S(t) + \psi G(t) S(t) + \phi G(t) \\ &= (\gamma + \varphi) G(t) + (\psi - \beta) G(t) S(t) \end{aligned}$$

Taking the limit supremum, we obtain

$\lim_{t \to \infty} \sup(A(t) + S(t) + G(t)) \leq \frac{(\gamma - \delta)}{\eta}$, where, $\gamma = \min(\gamma_1, \gamma_2)$ and

$\eta \leq M \frac{(\gamma + \varphi) + (\psi - \beta)}{(\gamma - \delta)}, M \geq 0$ in Ω.

In the theoretical ecology, positivity and boundedness of the system establishes the biological well behaved nature of the system.

3.3 Stability Analysis:

The equilibrium point of the model (1)-(3), are $E_o(0,0,0)$ and $E^*(A^*, S^*, G^*)$

Lemma 3: The equilibrium point $E_o(0,0,0)$ is locally asymptotically stable, if all the eigen values of the Jacobian matrix are negative.

Proof: Let us consider, $F_1 = \frac{dA}{dt}$, $F_2 = \frac{dS}{dt}$ and $F_3 = \frac{dG}{dt}$

Then the Jacobian Matrix of the above equations can be written as,

$$J = \begin{bmatrix} \frac{\partial F_1}{\partial A} & \frac{\partial F_1}{\partial S} & \frac{\partial F_1}{\partial G} \\ \frac{\partial F_2}{\partial F} & \frac{\partial F_2}{\partial S} & \frac{\partial F_2}{\partial G} \\ \frac{\partial F_3}{\partial A} & \frac{\partial F_3}{\partial S} & \frac{\partial F_3}{\partial G} \end{bmatrix} = \begin{bmatrix} -\alpha & 0 & 0 \\ \alpha & -\beta G + \psi G - \gamma_1 & \varphi - \beta S + \psi S & - \\ 0 & \beta G - \psi G & -\varphi + \beta S - \psi S - \delta \end{bmatrix}$$

At the equilibrium point, the Jacobian matrix is

$$J_{(0,0,0)} = \begin{bmatrix} -\alpha & 0 & 0 \\ \alpha & -\gamma_1 & \varphi \\ 0 & 0 & -\varphi - \delta \end{bmatrix}$$

The characteristic equation is $\left| J - \lambda I \right| = \begin{vmatrix} -\alpha - \lambda & 0 & 0 \\ \alpha & -\gamma_1 - \lambda & \varphi \\ 0 & \beta + \delta & -\varphi - \delta - \lambda \end{vmatrix} = 0$

Thus, $\lambda_1 = -\alpha$, $\lambda_2 = -\gamma_1$ and $\lambda_3 = -\varphi - \delta$

Since, λ_1, λ_2 and λ_3 are all negative, Hence, the equilibrium point E_0 is asymptotically stable .

Lemma 4: The equilibrium point $E^*(A^*, S^*, G^*)$ is asymptotically stable, if λ_1 and $\lambda_2 < 0$, Otherwise, it is unstable.

Proof: Then the Jacobian Matrix of the above equations can be written as,

$$J\big|_{(A^*,S^*,G^*)} = \begin{bmatrix} -\alpha & 0 & 0 \\ \alpha & -\beta G^* + \psi G^* - \gamma_1 & \varphi - \beta S^* + \psi S^* \quad - \\ 0 & \beta G^* - \psi G^* & -\varphi + \beta S^* - \psi S^* - \delta \end{bmatrix}$$

$$\lambda_1 = \frac{\beta G^* - \psi G^* + \gamma_1}{-\varphi + \beta S^* - \psi S^* - \delta}, \lambda_2 = \frac{-\beta G^* + \psi G^* - \gamma_1}{\varphi - \beta S^* + \psi S^* + \delta} \text{ and } \lambda_3 = -\alpha$$

Hence, the equilibrium point $E^*(A^*, S^*, G^*)$ is asymptotically stable, if λ_1 and $\lambda_2 < 0$, Otherwise, it is unstable.

3.4 Mathematical Model with Control:

Since, our goal is to find the control strategies so to reduce the falling of groundwater level as well as to show the optimal planning and management of groundwater level declination. So instead of pumping of water, we use the river stream i.e we use a control $u_1(t)$ (the diverting river stream) and use the reservoir to reduce the falling rate of ground water, i.e we use another control $u_2(t)$ (reservoir i.e storage system).

Because if the user may use the water from river stream to neglect the pumping water system, then a huge amount of water may be reserved in the ground level.

Taking this control, we assume the following system of nonlinear DE.

$$\frac{dA}{dt} = \gamma_1 S(t) + \gamma_2 G(t) - \alpha A(t) - u_2(t) A(t) \tag{5}$$

$$\frac{dS}{dt} = \alpha A(t) - \beta G(t) S(t) - \gamma_1 S(t) + \psi G(t) S(t) + \phi G(t) - u_1(t) S(t) \tag{6}$$

$$\frac{dG}{dt} = \beta G(t) S(t) - \gamma_2 G(t) + (1 - \psi) G(t) S(t) - \varphi G(t) - \delta G(t) + u_1(t) S(t) + u_2(t) A(t) \tag{7}$$

We want to find the control strategies so that the overall cost of the storage system and diverting river stream is minimized while maximizing the groundwater level (i.e. reduced the groundwater level declination).

Thus, the objective functional is chosen to be

$$J(u_1, u_2) = \underset{u_1, u_2}{Min} \int_{t_i}^{t_f} -G(t) + B u_1(t)^2 + C u_2(t)^2 dt \tag{8}$$

Here the cost function is a nonlinear function of $u_1(t), u_2(t)$; we choose them quadratic cost function for convexity. The parameters $B_1, C_1 \geq 0$ represents the desired 'weights' on the achievement and systematic cost.

3.5 Analytic Solution for optimal control problem: (Pontryagin's maximum principle)

Let us consider an optimal control problem

$$J(u) = \underset{u}{Min} \int_{t_0}^{t_1} f(t, x(t), u(t)) dt$$

Subject to $\dot{x}(t) = g(t, x(t), u(t))$, $x(t_0) = x_0$ and $x(t_1)$ free.

If $u^*(t)$ and $x^*(t)$ are optima solution then $\exists$ adjoint variable $\lambda(t)$ such that $H\left(t, x^*(t), u^*(t), \lambda(t)\right) \leq H\left(t, x^*(t), u(t), \lambda(t)\right)$ for all controls u at each time t, where the Hamiltonian is defined by, $H\left(t, x(t), u(t), \lambda(t)\right) = f\left(t, x(t), u(t)\right) + \lambda(t) g\left(t, x(t), u(t)\right)$

and $\lambda'(t) = \dfrac{-\partial H\left(t, x(t), u(t), \lambda(t)\right)}{\partial x}$, $\lambda(t_1) = 0$, $\dfrac{\partial H}{\partial u} = 0$ at u^* for each t [8].

In this problem, we introduce three adjoint variables $\lambda_1, \lambda_2, \lambda_3$ then the Hamiltonian

$$\begin{aligned} H\left(t, x(t), u(t), \lambda(t)\right) = & -G(t) + Bu_1(t)^2 + Cu_2(t)^2 + \lambda_1\{\gamma_1 S(t) + \gamma_2 G(t) - \alpha A(t) - u_2(t) A(t)\} \\ & + \lambda_2\{\alpha A(t) - \beta G(t) S(t) - \gamma_1 S(t) + \psi G(t) S(t) + \phi G(t) - u_1(t) S(t)\} + \lambda_3\{\beta G(t) S(t) \\ & - \gamma_2 G(t) + (1-\psi) G(t) S(t) - \varphi G(t) - \delta G(t) + u_1(t) S(t) + u_2(t) A(t)\} \end{aligned}$$

Where the variables $\lambda_i,\ i = 1, 2, 3$ are adjoint variables or costate variables which satisfy the following adjoint equations

$$\dot{\lambda}_1(t) = -\frac{\partial H}{\partial A} = \lambda_1(\alpha - u_2(t)) + \alpha\lambda_2 + \lambda_3 u_2(t)$$

$$\dot{\lambda}_2(t) = -\frac{\partial H}{\partial S} = \lambda_1\gamma_1 + \lambda_2\{\beta G(t) - \gamma_1 + \psi G(t)) - u_1(t)\} + \lambda_3\{\beta G(t) + (1-\psi) G(t) + u_1(t)\}$$

$$\dot{\lambda}_2(t) = -\frac{\partial H}{\partial G} = -1 + \lambda_1\gamma_2 + \lambda_2\{\beta S(t) + \psi S(t)) + \varphi\} + \lambda_3\{\beta S(t) + (1-\psi) S(t) - \gamma_2 - \varphi - \delta\}$$

with the transversality condition $\lambda_i(t_f) = 0, i = 1, 2, 3$

Theorem 1: For the optimal control problem (8), there exist an optimal control (u_1^*, u_2^*) that minimizes the objective function J over u such that, $u_1^* = \min\{1, \max(0, u_1)\}$ and $u_2^* = \min\{1, \max(0, u_2)\}$

Proof: We proved this theorem by considering the following cases

Case-I: i) $\dfrac{\partial H}{\partial u_1} < 0$, ii) $\dfrac{\partial H}{\partial u_1} = 0$, iii) $\dfrac{\partial H}{\partial u_1} > 0$,

Case-II: i) $\dfrac{\partial H}{\partial u_2} < 0$, ii) $\dfrac{\partial H}{\partial u_2} = 0$, iii) $\dfrac{\partial H}{\partial u_2} > 0$,

where, $\dfrac{\partial H}{\partial u_1} = 2Bu_1 - \lambda_2 S(t) + \lambda_3 S(t)$ and $\dfrac{\partial H}{\partial u_2} = 2Cu_2 - \lambda_1 A(t) + \lambda_3 A(t)$

4. Numerical Simulations

The model system is simulated using ode45 solvers written in MATLAB programming language. We use some of the parameter values from the following table, and by considering the initial conditions,
$A_0(t) = 5\times10^3, S_0(t) = 2.4\times10^4, G_0(t) = 10^4$.

Table 1: Values and Explanation of parameters

Descriptions	Symbols	Values
Precipitation rate	α	0.094
Infiltration rate	β	0.042
Evaporation rate from surface	γ_1	0.0056
Evaporation rate from Ground level	γ_2	0.0038
Deforestation rate	δ	0.063
Pumping rate of water	φ	0.079
Pollution rate	ψ	0.005

Again, using Forward-Backward Sweep Method [8], we solve the optimal control problem numerically. The numerical optimal solution of the state equations and adjoint equations with objective be found in MATLAB (R2019a) with weight parameters $B = 15000$ and $C = 25000$.

Firstly, we discuss the solution of the system of nonlinear ordinary differential equations when both controls are set to zero that means taking no policy. We see that the Ground water level is falling significantly. The result obtained for the equations (1)-(3) ,which are presented in Figure 2

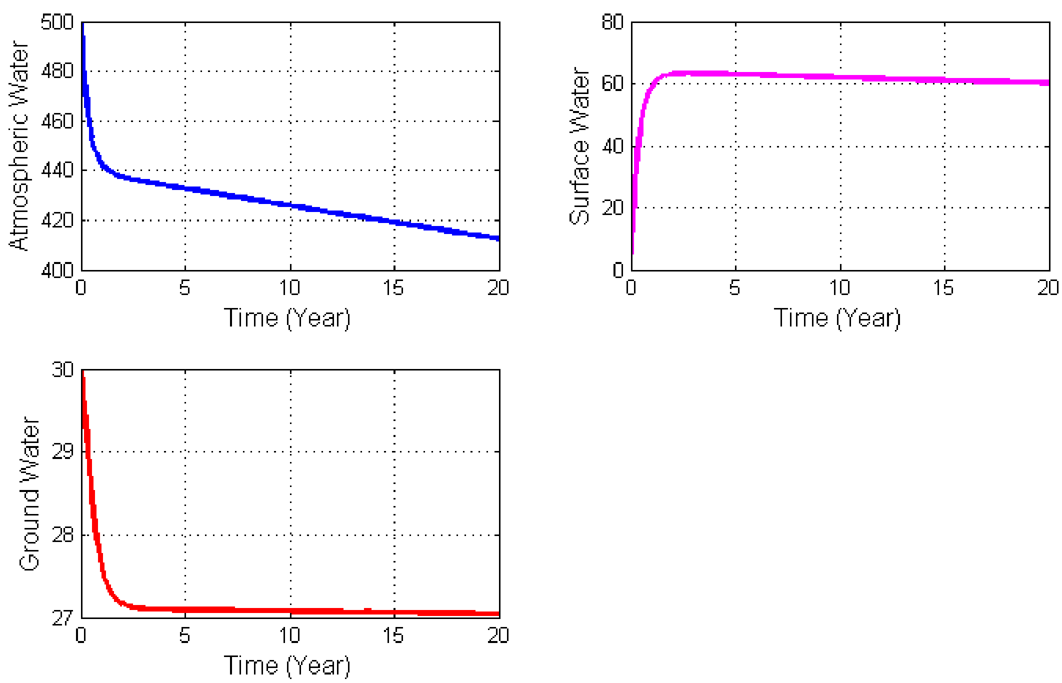

Figure 2: Numerical simulation for Groundwater model, with time (20 years)

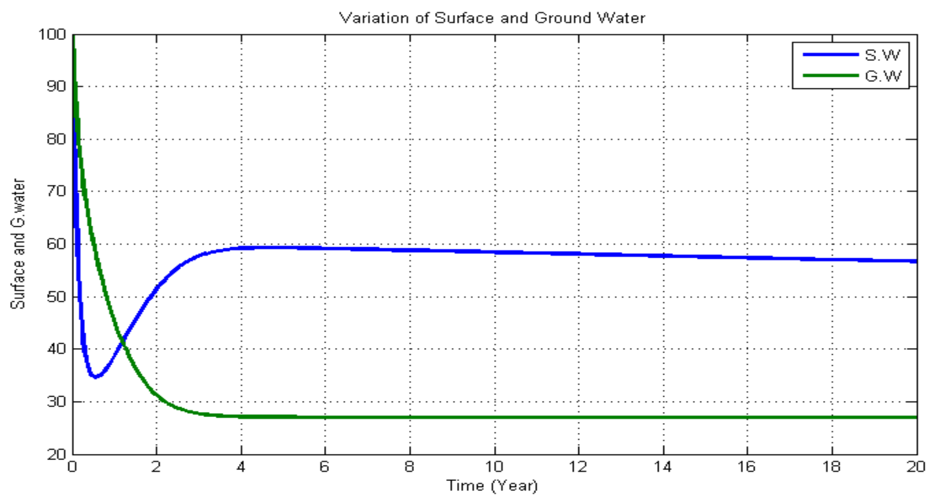

Figure 3: Variation of surface water and Ground water level

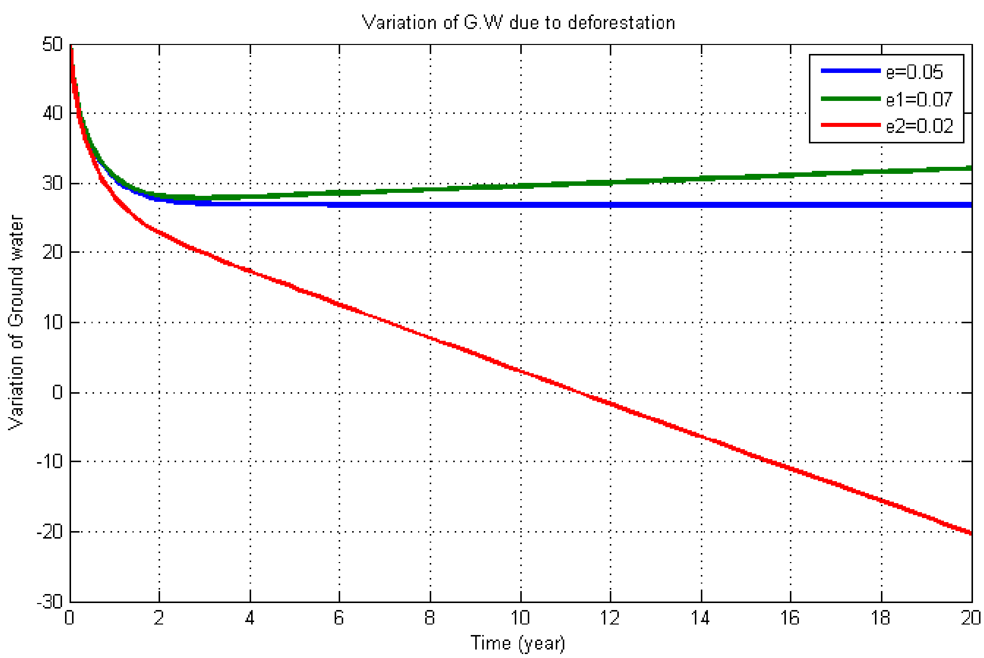

Figure 4: Falling of Groundwater level is significantly increased due to increase of deforestation rate

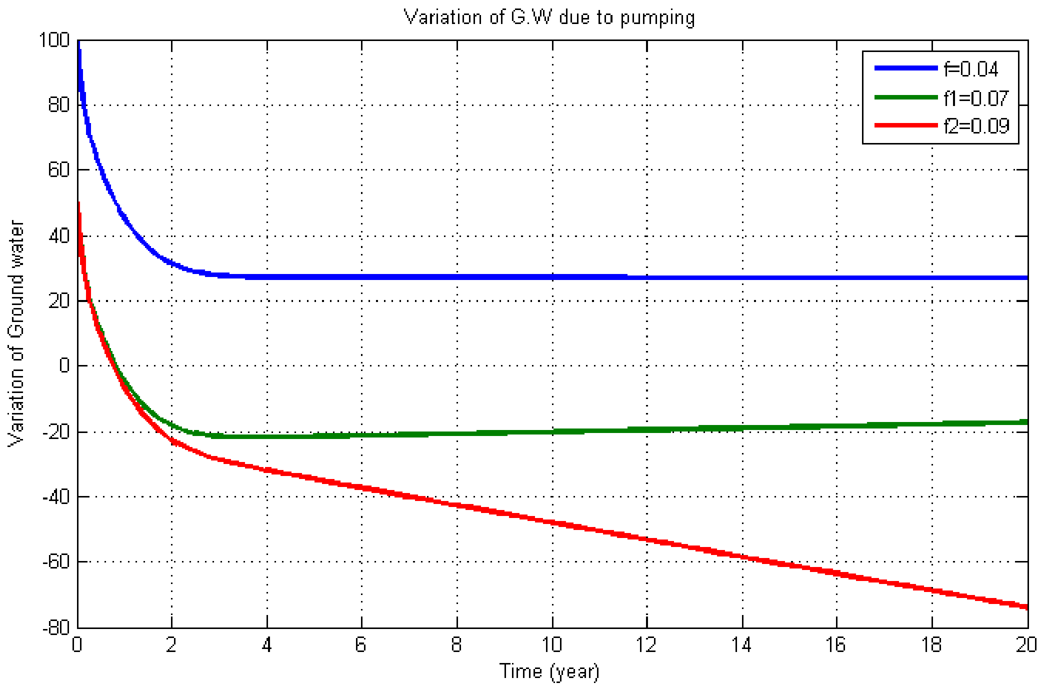

Figure 5: Falling of Groundwater level is significantly increased due to increase of pumping of water

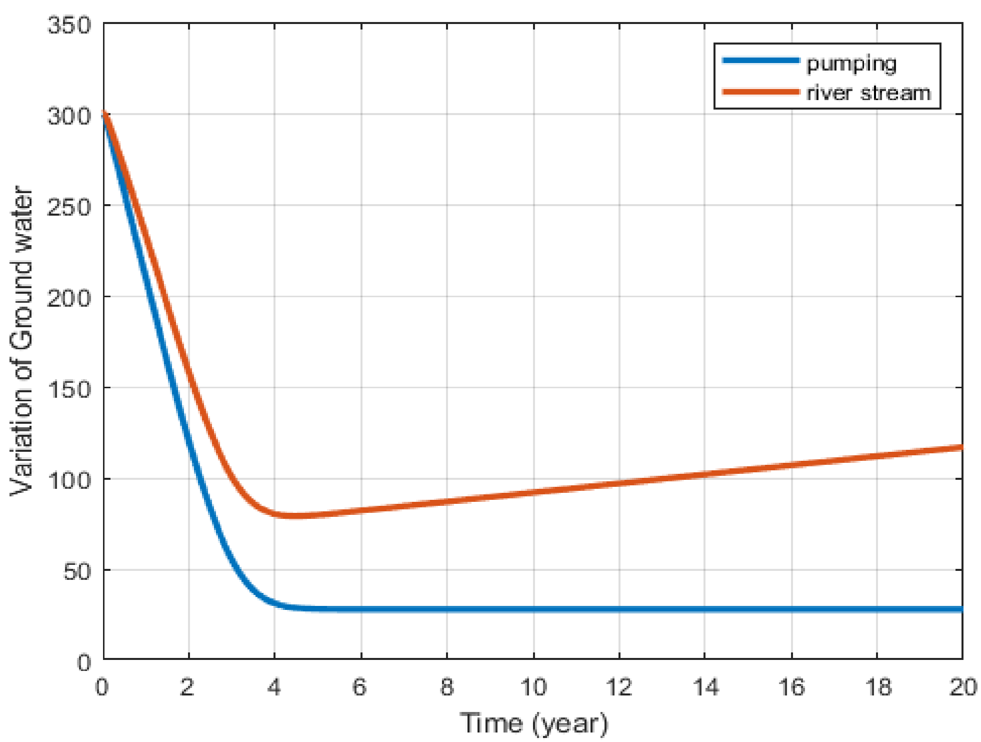

Figure 6: Groundwater level is significantly increased due to use of river stream instead of pumping

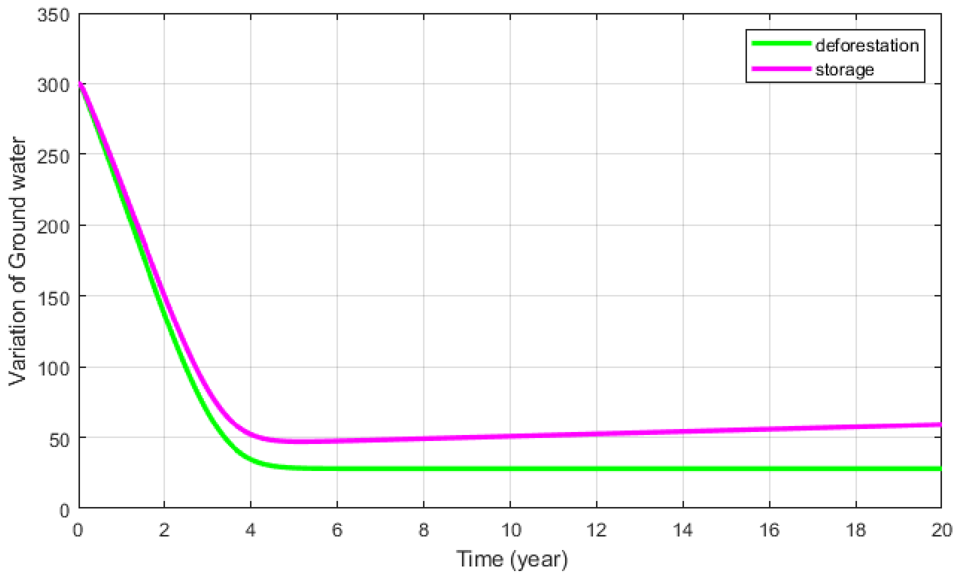

Figure 7: Groundwater level is increased due to use of reservoir to storage water

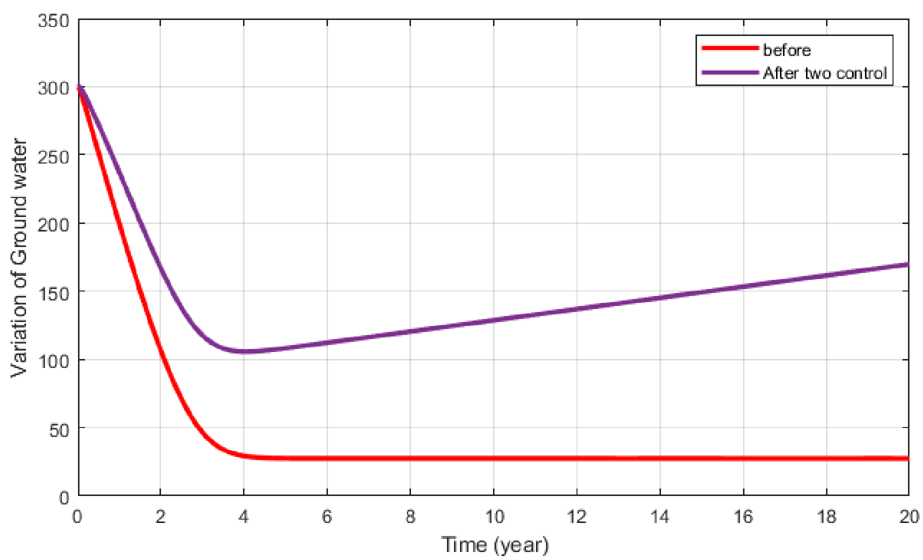

Figure 8: Groundwater level is significantly increased due to use of river stream and storage water i.e after using control.

5 Conclusions:

In this study, a mathematical model on Groundwater level declination introducing two control strategies diverting river stream and using storage system, to reduce the falling of ground water level with maximization of objective functional using Pontryagins maximum principle is presented. After successful implementation of the policy, it is clear that the value of objective functional is maximized with the minimization cost to divert river stream and to build up the water reservoir. We decide that the control strategies is more effective to increase the ground water level than frequent pumping of water from Ground level. It gives a latest picture of groundwater level management in Bangladesh as well as all over the world. The proposed model can be of help for the researchers and Planners who are associated with the research of groundwater level. It may be helpful for the Government to make and take decision regarding the prevention of groundwater level declination as well as may be increase the public awareness in case of using groundwater.

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