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Ranking Alternatives in Multiple Criteria Decision Analysis Based on a Common-Weight DEA

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Abstract

Ranking alternatives is an important issue in multiple criteria decision analysis (MCDA); especially that different approaches produce different results. This paper proposes a measure of relative distance, which involves the calculation of the relative position of an alternative between the anti-ideal and the ideal for ranking. In this case, minimizing the distance to the ideal is equivalent to maximizing the distance to the anti-ideal, so the rankings obtained from the two criteria are the same.

Keywords
Multiple criteria decision analysis, compromise programming, data envelopment analysis

1. Introduction
Ranking a group of alternatives based on a set of criteria frequently occurs in the real world. The associated research falls into the category of Multiple Criteria Decision Analysis (MCDA). Numerous MCDA methods for ranking alternatives have been developed [1,2]. The essence of each method is the way that the performances of the selected criteria are aggregated. Once the importance of each criterion is decided, the aggregate scores are calculated and the rankings are determined.

In this sense, the most critical step is determining the importance of each criterion. Usually, an ideal alternative is necessary to serve as a benchmark for comparing all alternatives, and the one that is closest to the ideal is preferred. Some studies [3,4] have discussed the idea that being farther away from the negative ideal, or anti-ideal, is better, where the negative ideal is the imaginary alternative which has the smallest value in each criterion. The alternatives are ranked based on their distance to the ideal or anti-ideal.

For two alternatives with the same distance to the ideal, the one which is farther away from the anti-ideal is considered better because it is “relatively” closer to the ideal. Similarly, for two alternatives with similar distances to the anti-ideal, the one which is closer to the ideal is preferred. In this regard, a measure of relative distance which shows the relative position of an alternative from the anti-ideal to the ideal is desirable. This paper formulates the problem of weight determination using a compromise programming technique, where the difference between the performances of the alternative and the ideal is treated as the distance. The rankings of the alternatives are based on the aggregate performance calculated from the set of weights. One attractive feature of the relative distance measure is that the rankings obtained based on the distance to the ideal and those obtained based on the distance to the anti-ideal are the same.

2. Graphical Illustration
In multiple criteria analysis, there will usually be several alternatives which are not dominated by the others. One of the nondominated alternatives is chosen for implementation. Charnes et al. [5] proposed the DEA technique to calculate the relative efficiency of a group of decision making units (DMUs) which uses multiple inputs to produce multiple outputs. Each unit is allowed to use different sets of weights to calculate the efficiency. Those with an efficiency value of 1 are nondominated units. The MCDA problem can be considered as a DEA problem without inputs. Hence, the DEA technique can be applied to identify nondominated alternatives.

Let Y_{ij} denote the value of the ith criterion, i=1,…, m, for the jth alternative, j=1,…, n. The DEA model without inputs for calculating the efficiency of the kth alternative can be formulated as [6]:

$$\begin{aligned} E_k = \max\ & \textstyle\sum_{i=1}^{m} w_i Y_{ik} \\ \text{s.t.}\ & \textstyle\sum_{i=1}^{m} w_i Y_{ij} \le 1, \quad j = 1,\ldots, n \\ & w_i \ge \varepsilon, \qquad\qquad i = 1,\ldots, m \end{aligned} \tag{1}$$

where w_i is the importance associated with the ith criterion and ε is a small positive quantity imposed to restrict any criterion from being ignored. The most favorable weights are sought for each DMU in calculating its efficiency. This model has a dual which is exactly the same as the output-oriented BCC model without inputs [7].

Suppose there are five alternatives, A, B, C, D, and E. Their performances in two criteria, Y_1 and Y_2, are shown in Table 1 and are depicted in Figure 1. Model (1) identifies B and D as the nondominated alternatives. The piecewise line segments $SBDT$ represent the efficiency frontier constructed from the five alternatives. Alternative E lies on the vertical line extended downward from the efficient alternative D, and is called weakly efficient. Alternatives A and C lie in the interior of the area delineated by line segments $SBDT$ and are thus dominated. Their efficiency values, as calculated from Model (1), are the ratios of OA to OA' and OC to OC', respectively, where A' and C' are the projections on their respective frontier facets. Column 4 of Table 1, with the heading "DEA efficiency", shows the efficiency values of the five alternatives. Numbers in parentheses are their ranks.

In this case, B and D are the candidates for selection because they are nondominated. However, it is not clear which one is better. Alternative E is ranked the next best since it is weakly efficient. For the two dominated alternatives, A and C, C is better due to its higher efficiency value. Geometrically, the calculation of the efficiency value is based on the frontier facet with which the alternative is compared. Since the efficiency value is generally determined from different frontier facets, different weights are applied in calculating the efficiency. Hence, the alternatives should not be ranked simply by their efficiency values [8].

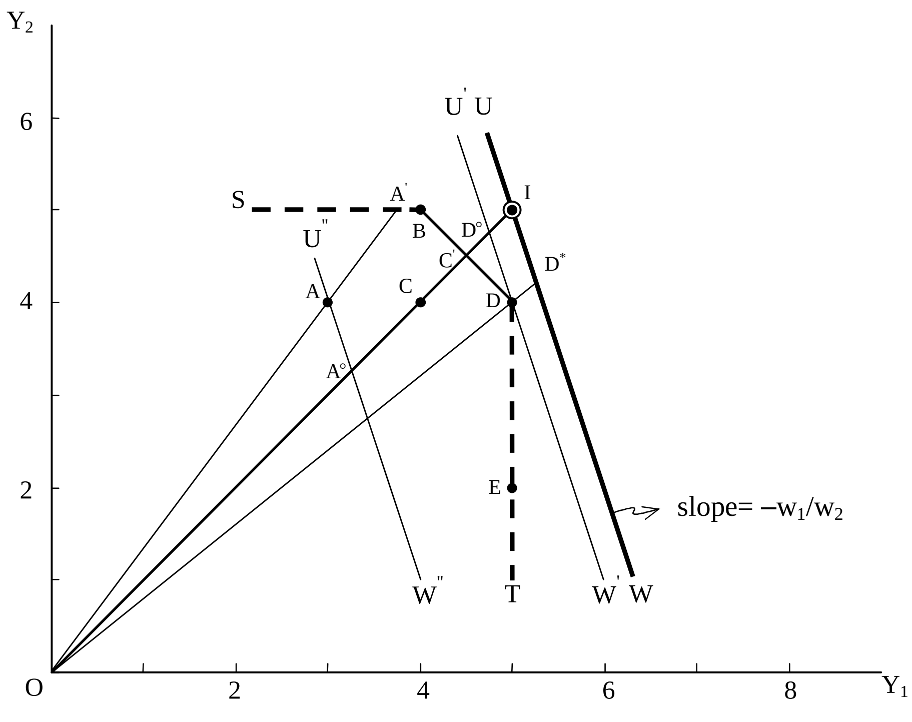

Figure 1: Geometric interpretation of the relative performance compared with the ideal, *I*

Table 1: Data and various performance measures for the given example

Alternative	Y_1	Y_2	DEA efficiency	Absolute distance		Relative distance	
				Ideal	Anti-ideal	Ideal	Anti-ideal
A	2	4	0.8 (5)	$(17)^{1/2}/15$ (5)	$(37)^{1/2}/15$ (5)	0.35 (5)	0.65 (5)
B	4	5	1 (1)	$(4)^{1/2}/15$ (2)	$(64)^{1/2}/15$ (4)	0.15 (2)	0.85 (2)
C	4	4	8/9 (4)	$(5)^{1/2}/15$ (3)	$(65)^{1/2}/15$ (3)	0.20 (4)	0.80 (4)
D	5	3	1 (1)	$(1)^{1/2}/15$ (1)	$(101)^{1/2}/15$ (1)	0.05 (1)	0.95 (1)
E	5	2	1–ε (3)	$(9)^{1/2}/15$ (4)	$(100)^{1/2}/15$ (2)	0.15 (2)	0.85 (2)

3. Absolute Distance

The first task in calculating the distance between an alternative and the ideal or the anti-ideal is to find the ideal and the anti-ideal. The anti-ideal, which has the smallest value in all criteria, is easier to determine. The origin is the theoretical anti-ideal because it has a value of zero for every criterion. The ideal, on the other hand, is difficult to identify because not every criterion has a theoretical ceiling.

Consider again a set of *n* alternatives with *m* criteria. The performance of alternative *j* in criterion *i* has a value of Y_{ij}. Let Y_i^* = max $\{Y_{ij}, j=1,\ldots, n\}$ denote the largest value that appears in the *i*th criterion. Then, $I=(Y_1^*, Y_2^*,\ldots, Y_m^*)$ is empirically the ideal alternative. For example, the ideal alternative generated from the five alternatives in Table 1 is I=(5, 5). By the same token, if an empirical, rather than the theoretical, anti-ideal is preferred, then one can define $I^-=(Y_1^-,\ldots, Y_m^-)$ as the anti-ideal, where Y_i^- = min$\{Y_{ij}, j=1,\ldots, n\}$ is the smallest value that appears in the *i*th criterion.

One of the most popular approaches for ranking alternatives is compromise programming, which is based on the distance between the alternative and the ideal. Alternatives with a shorter distance to the ideal are considered better than those with a longer distance. Usually a weight is applied to the values of each criterion to make all criteria comparable. When no prior information is available, one can use the information contained in the observations to generate the weight.

Consider the example in Table 1. Let w_1 and w_2 be the weights of criteria Y_1 and Y_2, respectively. The squared distance between alternative A=(3, 4) and the ideal I=(5, 5) is $[w_1(5–3)]^2+[w_2(5–4)]^2=4w_1^2+w_2^2$. Calculating the squared distance for the other four alternatives results in a total squared distance of $6w_1^2+12w_2^2$. For general cases, the total squared distance from all alternatives is $\sum_{j=1}^n[\sum_{i=1}^m w_i^2(Y_i^*-Y_{ij})^2]$. To exclude the trivial solution of w_i^*=0, one can require the aggregate performance of the ideal alternative to have a value of 1. Thus, rather than assigning the weights beforehand, they are obtained by minimizing the total squared distance to the ideal:

$$\begin{aligned} \min\ & \textstyle\sum_{j=1}^n\sum_{i=1}^m w_i^2(Y_i^*-Y_{ij})^2 \\ \text{s.t. } & \textstyle\sum_{i=1}^m w_iY_i^*=1 \\ & w_i\geq\varepsilon,\quad i=1,\ldots,m \end{aligned} \tag{2}$$

The small quantity ε is introduced so that no criterion is ignored.

Model (2) is a quadratic program. After the optimal weights w_i^*, i=1,…, m, are solved, the distance between each alternative and the ideal can be calculated; the ranks of the alternatives can then be determined. Using the data in Table 1 as an example, the optimal weights are w_1^*=2/15 and w_2^*=1/15. The ratio of the two weights is w_1/w_2=2, indicating that the scale of the first criterion must be doubled to make the two criteria comparable. Using this set of weights, the distance from each alternative to the ideal, $[\sum_{i=1}^{m} w_i^2 (Y_i^* - Y_{ij})^2]^{1/2}$, is calculated as shown in the fifth column of Table 1. The corresponding ranks appear in parentheses.

Compared with the results of DEA-efficiency, the nondominated alternative *D* is also ranked first with the absolute-distance approach. The other nondominated alternative *B* is the second best. The weakly efficient alternative *E* has a rank of 4, instead of 3 as in the DEA-efficiency approach. The fourth ranked alternative, *C*, is ranked third. The fifth ranked alternative, *A*, remains the same. Three of the five alternatives have different ranks, which is due to every alternative in the DEA-efficiency approach possible using different sets of weights for comparison, while the absolute-distance approach requires all alternatives to use the same set of weights. According to Adler et al. [8], results from different sets of weights are not suitable for ranking.

In compromise programming, alternatives are ranked according to their distance to the ideal or to the anti-ideal. The alternative which is closest to the ideal need not be the same as that farthest away from the anti-ideal. In this example, the origin is the theoretical anti-ideal alternative because it has the smallest value in both criteria. Using the weights of w_1^* =2/15 and w_2^* =1/15 obtained from Model (2), the distances to the anti-ideal for the five alternatives are $(37)^{1/2}/130$, $(64)^{1/2}/130$, $(65)^{1/2}/130$, $(101)^{1/2}/130$, and $(100)^{1/2}/130$, respectively. The corresponding ranks are 5, 4, 3, 1, and 2, respectively, as shown in the sixth column of Table 1. Alternative *B* and *E* have ranks different from those obtained from the distance to the ideal. Compared to the DEA-efficiency approach, there are three alternatives whose ranks are different.

The reason for obtaining different rankings is simply that the absolute-distance approach only considers the distance to the ideal, disregarding the distance to the origin. If one can find a distance measure which takes both the ideal and anti-ideal into consideration, then consistent rankings may be obtained.

4. Relative Distance

The aggregate performance of any alternative is worse than that of the ideal, no matter what weights are applied to individual criteria. Thus, we have $P_j = \sum_{i=1}^{m} w_i Y_{ij} / \sum_{i=1}^{m} w_i Y_i^* < 1$, where P_j is the aggregate performance of the *j*th alternative relative to the ideal. The ideal alternative has a relative performance value of 1: $P^* = \sum_{i=1}^{m} w_i Y_i^* / \sum_{i=1}^{m} w_i Y_i^* = 1$. If we let $\sum_{i=1}^{m} w_i Y_i^*$, which is the aggregate performance of the ideal alternative, be equal to 1 to standardize the weights w_i, then $\sum_{i=1}^{m} w_i Y_{ij}$ becomes the relative performance of the *j*th alternative. The difference between P_j and 1, denoted by s_j, is the relative distance between the *j*th alternative and the ideal in terms of the aggregate performance. It is also the complementary performance of this alternative. The problem is then transformed to finding the set of weights w_i, i=1,…, m, which produce the smallest total squared difference between the relative performance of the alternative and that of the ideal. The associated model is:

$$\begin{aligned} \min\ & \textstyle\sum_{j=1}^{n} s_j^2 \\ \text{s.t.}\ & \textstyle\sum_{i=1}^{m} w_i Y_{ij} + s_j = 1, \quad j = 1,\ldots,n \\ & \textstyle\sum_{i=1}^{m} w_i Y_i^* = 1 \\ & w_i \geq \varepsilon, \quad i = 1,\ldots,m \end{aligned} \tag{3}$$

Note that the distance variable s_j is always positive because every alternative $\boldsymbol{Y}_j$ is dominated by the ideal $I=\boldsymbol{Y}^*$. After the optimal weights w_i^*, i=1,…, m, are obtained, the relative performance of the *j*th alternative is calculated as $\sum_{i=1}^{m} w_i^* Y_{ij}$. The relative distance to the ideal is $s_j^* = 1 - \sum_{i=1}^{m} w_i^* Y_{ij}$.

Comparing Model (3) with the conventional DEA model without inputs, i.e., Model (1), it is noted that the constraints of the two models are essentially the same, except that Model (3) has an extra constraint associated with the ideal alternative. In the context of DEA, the ideal alternative is also included to construct the production frontier. The difference between the two models is the objective function; Model (1) maximizes the aggregate performance of each specific alternative in each calculation, while Model (3) minimizes the total squared complementary performance, or the total squared relative distance to the ideal, of all alternatives in one calculation. The weights used by each alternative in calculating the aggregate performance can be different in Model (1); they are the same in Model (3). They are the general consensus of the alternatives being evaluated. The same set of weights provides a common base for comparing different alternatives.

Geometrically, the frontier constructed by Model (1) is a set of connected facets, while that constructed by Model (3) is a single-facet hyperplane, $\sum_{i=1}^{m} w_i Y_i = 1$, passing through the ideal. Note that here w_i's in $\sum_{i=1}^{m} w_i Y_i = 1$ are constants and Y_i's are coordinates. Model (3) can be considered as a common-weight DEA model [9]. The hyperplane $\sum_{i=1}^{m} w_i Y_i = 1 - s_j$, which passes through alternative j, is parallel to the frontier $\sum_{i=1}^{m} w_i Y_i = 1$, with a distance of s_j. Since s_j represents the relative position of alternative j from the origin to its projection on the frontier, it is a relative distance measure. Substituting s_j in the objective function of Model (3) by $(1 - \sum_{i=1}^{m} w_i Y_{ij})$, or $(\sum_{i=1}^{m} w_i Y_i^* - \sum_{i=1}^{m} w_i Y_{ij})$, from the constraints and omitting the first set of constraints, Model (3) can be simplified to:

$$\begin{aligned} \min\ & \textstyle\sum_{j=1}^{n} [\sum_{i=1}^{m} w_i (Y_i^* - Y_{ij})]^2 \\ \text{s.t.}\ & \textstyle\sum_{i=1}^{m} w_i Y_i^* = 1 \\ & w_i \geq \varepsilon, \quad i = 1, \ldots, m \end{aligned} \tag{4}$$

The relative distance to the ideal, $\sum_{i=1}^{m} w_i (Y_i^* - Y_{ij})$, is the basis for ranking.

The aggregate performance, $\sum_{i=1}^{m} w_i Y_{ij}$, represents the relative distance of alternative j to the origin. Larger values imply a location farther away from the anti-ideal. Since $\sum_{i=1}^{m} w_i Y_{ij}$ is the complement of s_j, the alternative with the smallest distance to the ideal, s_j, obviously has the largest distance to the anti-ideal, $\sum_{i=1}^{m} w_i Y_{ij}$. Hence, the relative distances of an alternative to the ideal and to the anti-ideal produce the same rankings.

For cases where the origin is not suitable to be the anti-ideal, and the empirical anti-ideal, $I^- = (Y_1^-, Y_2^-, \ldots, Y_m^-)$, is preferred, then P_j, the aggregate performance of the jth alternative relative to the ideal, can be adjusted by the aggregate performance of the anti-ideal as: $P_j = \sum_{i=1}^{m} w_i (Y_{ij} - Y_i^-) / \sum_{i=1}^{m} w_i (Y_i^* - Y_i^-)$. The geometric meaning is a translation of the origin to I^-. In this case, the adjusted performance of the ideal, $\sum_{i=1}^{m} w_i (Y_i^* - Y_i^-)$, is set to 1 to standardize the weight w_i. Furthermore, since the scale of Y_{ij} could be very large or very small, which would make the lower bound ε in $w_i \geq \varepsilon$ difficult to determine, a relative bound, $w_i (Y_i^* - Y_i^-) \geq b$, which requires the contribution of each criterion to the aggregate performance to be greater than a proportion b, is recommended. It is easy to prove that, with this adjustment, the obtained weights produce the same rankings for the criterion of "closer to the ideal is better" and that of "farther away from the anti-ideal is better".

Following Model (4), the optimal weights obtained for the data in Table 1 are $w_1^* = 0.15$ and $w_2^* = 0.05$. The frontier for calculating the aggregate performance of each alternative is a straight line, $0.15Y_1 + 0.05Y_2 = 1$, passing through the ideal with a slope of $-w_1/w_2 = -3$. The relative distance to the ideal for alternative j, s_j, is the ratio of the distance between the alternative and its projection on the frontier to the distance between the origin and the projection point on the frontier. Its complement, $1 - s_j = \sum_{i=1}^{m} w_i Y_{ij}$, is the relative distance to the anti-ideal, which is also the relative performance value of this alternative. The last two columns of Table 1 show the relative distances of the five alternatives to the ideal and anti-ideal, respectively, with their ranks parenthesized. As expected, the rankings based on these two distance measures are exactly the same.

In Figure 1, every alternative is compared with its projection on the frontier, UW, in calculating the aggregate performance. For example, the performance value of D is the ratio of OD to OD^*, where D^* is the projection of D on the frontier. DD^*/OD^* is the relative distance to the ideal and OD/OD^* is the relative distance to the anti-ideal. Let $U'W'$ be a straight line passing through D and parallel to the frontier UW. This line is represented by $0.15Y_1 + 0.05Y_2 = 1 - s_D$. Suppose that the line connecting the origin and the ideal point intersects line $U'W'$ at D^o. It can be shown that $OD/OD^* = OD^o/OI$. Since the length of OI has been rescaled to 1, OD/OD^* is equal to the length of OD^o, or w_1Y_{1D} + w_2Y_{2D}, which is the aggregate performance of D. Thus, comparing an alternative with the ideal is equivalent to comparing it with its projection on the frontier. Similarly, one can draw a parallel line U"W", $0.15Y_1+0.05Y_2 = 1- s_A$, for alternative A, which intersects line OI at A^o. The length of OA^o is the aggregate performance of alternative A, with a value of $w_1Y_{1A} + w_2Y_{2A}$. For the general case of n alternatives, n parallel hyperplanes are constructed; each has a distance of s_j to the frontier. The alternative with the shortest distance to the frontier has the highest rank.

For a set of weights $\hat{w}_i$, i=1,…, m, "$\sum_{i=1}^{m} \hat{w}_i Y_i^* = 1$" represents a hyperplane passing through the ideal I. Following the above discussion, the relative performance of an alternative is equal to the ratio of the distance between the origin and the alternative to that between the origin and the projection of the alternative on the hyperplane. This value is equal to $1- s_j$, the complement of the distance between the parallel hyperplanes passing through the alternative and the ideal. Therefore, this set of weights produces the same rankings for the criteria of "closer to the ideal is better" and "farther away from the anti-ideal is better".

5. Conclusion

The conventional idea of seeking the shortest absolute distance between the alternative and the ideal in MCDA may produce results which are different from those obtained by seeking the longest absolute distance between the alternative and the anti-ideal. This paper showed that when the measure of distance is changed from absolute to relative, that is, the relative position of the alternative between the anti-ideal and the ideal, then the resultant rankings from the two ideas are consistent.

Since the weights used for calculating the aggregate performance for the alternatives in the proposed method are the most favorable for all alternatives in a compromise sense, the resultant rankings are convincing. Moreover, the weights are not subjectively determined by humans, which sometimes creates controversy; hence, the results are more acceptable when the alternatives are people or organizations.

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