Optimal Production Run Length for Products Sold with Warranty in a Deteriorating Production System with a Time Varying Defective Rate under Allowable Shortages

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Abstract

The paper studies the optimal production run length for a deteriorating production system. The optimal warranty cost is determined by a balance between the warranty cost of the manufacturer and the buyer during its warranty period. It is assumed that the elapsed time until the production process shift is age dependent and arbitrarily distributed. The objective is to minimize the total cost per item for determining the optimal production run length and the length of time when backorder is replenished. It shows that there exists a unique optimal production run length to minimize the expected total cost.

Keywords
Imperfect production processes, inventory, warranty period, time varying, production run length, shortages.

1. Introduction
Traditionally, an optimal production run length can be obtained by determining an economic manufacturing quantity (EMQ) model, for example, see Johnson and Montgomery [1]. The classical EMQ model assumes that the production process is perfect and stationary. However, this assumption may not be valid in practice. Usually, a nonconforming item incurs more post-sale servicing cost than a conforming item, especially when the products are sold with warranty. To improve quality as to reduce the warranty cost, one should shorten the production run length to allow for frequent restoration, thereby retaining the system into an in-control state. However, a short production run length results in more setup cost or restoration cost. Therefore, there is a need to determine an optimal production run length so that the expected total cost per item is minimized.

The key feature differentiating this paper from the extensive literature, is that in the out-of-control state, the expected number of nonconforming items is time dependent. As the time of the out-of-control period increases, the number of nonconforming items increases and the warranty period becomes a decision variable. Chen and Lo [2] seeks to minimize the total cost per item through optimal determination of production run length (t) and the time length when back ordered is replenished for an imperfect production system with allowable shortages for products sold with free minimal repair warranty. The present paper aims to extend Chen and Lo's [2] model to consider the warranty period (*w*) as an additional decision variable.

2. Notations
The following notations are used.

K	:	Setup cost per cycle
Z	:	The maximum inventory level
D	:	Demand rate in units per unit time
P	:	Production rate in units per unit time; $p > d$
C	:	The material cost to produce an item
H	:	Holding cost per unit per unit time
S	:	Shortage cost per unit per unit time
R	:	Restoration cost per cycle
$Cw_{(m)}$	:	Cost incurred at each minimum repair during warranty the period for the manufacturer
$Cw_{(b)}$	:	Cost incurred at each minimum repair during the warranty period for the buyer
Cw	:	Total cost incurred during the warranty period per unit time

X	:	A random variable which is the elapsed time for the production process shifts to "out-of-control" in a production cycle
F(x)	:	Probability density function for X and is assumed to be exponentially distributed with the parameter λ
λ	:	The parameter for exponential distribution f(x)
N	:	The number of non conforming items
$\theta 1$	:	Probability of non conforming items when the production process is in the "in-control" state
$\theta 2$	:	Probability of non conforming items when the production process is in the "out-of-control" state; and $0< \theta 1< \theta 2$
h1(t)	:	Constant failure rate function of non conforming items when the production process is in the "in-control" state
h2(t)	:	Increasing failure rate function of non conforming items when the production process is in the "out-of-control" state
ν	:	Failure distribution shape parameter
$\alpha(t)$	:	The fraction of the nonconforming items
w	:	Warranty period for each cycle
$r1(\tau)$	:	Failure rate function of conforming items
$r2(\tau)$	:	Failure rate function of nonconforming items
T	:	Cycle time
T1	:	Production time when backorder is replenished
T2	:	Production time when inventory builds up
T3	:	Time period when no production and inventory depletes
T4	:	Time period when no production and shortages occur
T	:	Production run length, where t = T1 + T2 and T = T1 + T2 +T3 +T4
T*	:	Optimal production run length for each cycle
TCPI	:	Total cost per item

3. Assumptions

The following assumptions are made

- The production cycle starts in an in-control state, producing perfect items.
- Any time point in the imperfect production cycle can be classified as an "in-control" state or "out-of-control" state. In the in-control state, a constant percentage of nonconforming items are produced. However, in the case of the out-of-control state, the percentage of nonconforming items is an increasing function of time, it depends on how much time it is out-of-control.
- The shift to the "out-of-control" state cannot be detected until the end of the production cycle. If the production process is in the "out-of-control" state, then it will be brought to the "in-control" state with a restoration cost for the following production cycle.
- The demand rate is deterministic and constant with allowable backordered shortages.
- The free minimal repair warranty (FRW) policy is adopted to formulate the problem.

Under the above assumptions, the graphic representation of the inventory behavior for the imperfect production system for the products sold with warranty can be shown as Figure 1.

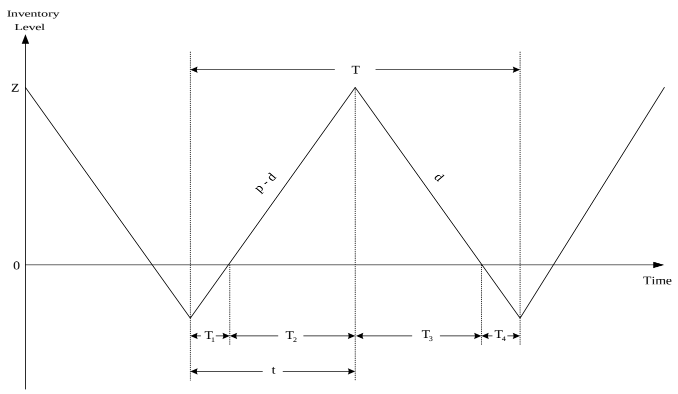

Figure 1: Graphic representation of the inventory behavior for the imperfect production system for the products sold with warranty.

To model the problem, the inventory cycle is divided into four major phases: backorder replenishment period (phase 1), inventory building period (phase 2), inventory depletion period (phase 3) and shortage period (phase 4). Time for phase *i* is indicated by T_i.

4. Mathematical model

The deterioration of a production system at any point in time can be classified as either - in-control and out-of-control. It is assumed that the elapsed time, *X*, of the system in the in-control state follows an exponential distribution with mean 1/λ. Once the system shifts to the out-of-control state, it stays there until the end of a production run. After the completion of a production run, the system is setup with cost $K > 0$ and is inspected to reveal the state of the system. If the system is out-of-control, then it is brought back to the in-control state with an additional restoration cost $r > 0$ for the next production run. Suppose that the production rate of the system and the demand rate of the product are *p* and *d*, respectively, where $p > d > 0$. The manufacturing cost of an item is *cm* and the inventory holding cost for carrying a product is $h > 0$ per unit time. It is assumed that all items can be classified as either conforming or nonconforming. Let *h1*(t) and *h2*(t) denote the constant failure rate and increasing failure rate associated with a conforming and a nonconforming item, respectively. In the out-of-control state, the expected number of nonconforming items is time dependent. We further assume that *h1*(t) < *h2*(t) for $t \geq 0$ which implies a nonconforming item is more likely to fail than a conforming item. Due to manufacturing variability, an item is nonconforming with probability θ_1 ($\theta_2 \nu t^{\nu-1}$) when the production process is in-control (or out-of-control). Since a nonconforming item can only be detected after a period of time in use, all the items produced are released for sale with a free minimal repair warranty. Under the free minimal repair warranty, failures that occur within the warranty period w result in valid warranty claims and are rectified by minimal repairs instantaneously at no cost to the buyers. After a minimal repair, the failure rate of an item remains the same as that just before failure. However, each minimal repair incurs a cost of c to the manufacturer.

The overall cost for our problem includes production costs, inventory holding costs, shortage costs, restoration costs and warranty costs. The formulations of these five costs are described in detail as follows.

5.1 Production cost (PC)

The production cost (PC) per cycle is composed of the fixed setup cost *K* and the variable unit manufacturing cost *cpt*. Namely,

$$PC = K + cpt \tag{1}$$

5.2 Holding cost (HC)

During each cycle, holding cost will occur during T2 and T3.
We note that the maximum inventory level Z is equal to (*p* − *d*) (*t* − T1) and the time duration with positive inventory is $T_2 + T_3 = (t - T_1) + \frac{(p-d)(t-T_1)}{d} = \frac{p}{d(t-T_1)}$ and the total inventory per cycle is computed as follows

$$-\frac{1}{2}Z(T_2 + T_3) = \frac{1}{2}[(p-d)(t-T_1)]\frac{p(t-T_1)}{d} = \frac{p(p-d)(t-T_1)^2}{2d} \tag{2}$$

Therefore, the total inventory holding cost (HC) per cycle is obtained as equation (3)

$$HC = \frac{\boldsymbol{h}p(p-d)(t-T_1)^2}{2d} \tag{3}$$

5.3 Shortage Cost (SC)

The total shortage per cycle can be obtained as follows

$$\frac{1}{2}T_1(p-d)T_1 + \frac{1}{2}T_4 d T_4 = \frac{p(p-d)T_1^2}{2d} \tag{4}$$

Therefore, the total shortage cost (SC) per cycle is obtained as equation (5)

$$SC = s\frac{p(p-d)T_1^2}{2d} \tag{5}$$

5.4 Restoration cost (RC)

The restoration cost (RC) per cycle can be obtained as follows

$$RC = rProb(X < t) = r(1 - e^{-ut}) \tag{6}$$

5.5 Warranty cost (WC)

Before formulating the warranty cost per cycle, we have to find out the expected number of nonconforming items per cycle, E(N) and the fraction of nonconforming items in a production cycle, α(t). The process can have two types of hazard rates, either a constant hazard rate or a varying hazard rate. The number of nonconforming items N can be expressed as follows:

$$N = \begin{cases} h_1(t)pt & for \quad X \geq t \\ h_1(t)pX + h_2(t)p(t-X) & for \ X < t \end{cases} \quad (7)$$

Where 0 < h1 (t) < h2 (t) and h1 (t) = θ1 (constant hazard rate) and
h2 (t) = θ2 υ t υ-1 (increasing hazard rate i.e., time varying hazard rate)
The expected number of nonconforming items in a production cycle is then given by

$$E(N) = \int_t^{\infty} \theta_1 pt\lambda e^{-\lambda x} dx + \int_0^t (\theta_1 px + \theta_2 \nu t^{\nu-1} p(t-x)) f(x) \lambda e^{-\lambda x} dx \quad (8)$$

Moreover, the fraction of the nonconforming items in a production cycle, α (t), is given by

$$\alpha(t) = \frac{E(N)}{pt} = \theta_2 \nu t^{\nu-1} + \frac{(\theta_1 - \theta_2 \nu t^{\nu-1})(1 - e^{-\lambda x})}{\lambda t} \quad (9)$$

Under the policy of free minimal repair warranty, we note that the failure process of a conforming product follows a non-homogeneous process with failure rate function r_1 (τ) while the failure process of a nonconforming product follows a non-homogeneous process with failure rate function r_2 (τ). Therefore, given a conforming item, the probability that this conforming item fails within the warranty period *w* is $\int_0^w r_1(\tau)d\tau$. On the other hand, given a nonconforming item, the probability that this nonconforming item fails within the warranty period *w* is $\int_0^w r_2(\tau)d\tau$. Consequently, for any given product, the probability of this product to be a failure within the warranty period can be obtained as follows:

$$(1 - \alpha(t)) \int_0^w r_1(\tau)d\tau + \alpha(t) \int_0^w r_2(\tau)d\tau \quad (10)$$

The warranty cost C_w depends on the warranty period *w*. However, the warranty cost could be shared by both the manufacturer and the buyer. As the warranty period increases the cost to the manufacturer goes up and cost of the defective product for the buyer decreases. This phenomenon is illustrated in Figure 2. So it is natural to think for an optimal warranty period for which the total societal cost is a minimum. As such we propose the following cost function for the manufacturer and the buyer. For the manufacturer, warranty cost, $C_{w(m)} = ae^{bw}$ and for the buyer, warranty cost, $C_{w(b)} = ce^{-dw}$ are assumed, where *a*, *b*, *c* and *d* are positive constants.

Table 1: Optimal warranty period, manufacturer cost and buyer cost under different parameters

a	b	c	d	w*	$C_{w(m)}*$	$C_{w(b)}*$	C_w
3.245	0.065	17.25	0.065	12.852	7.4820	7.4815	14.9635
1.245	0.065	31.215	0.065	24.783	6.2341	6.2339	12.4680
0.525	0.065	34.625	0.065	32.223	4.2637	4.2635	8.5272

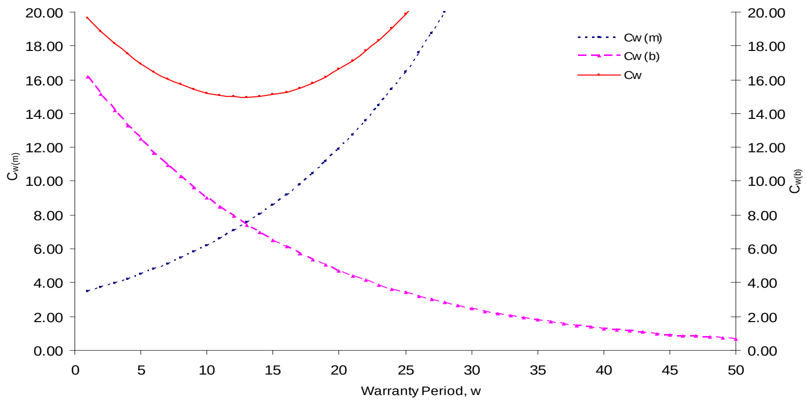

Figure 2: Manufacturer's and buyer's cost during warranty period w

Therefore, the warranty cost (WC) per cycle is given by

$$\text{WC} = \frac{pt}{w}\left(\int_0^w a e^{bw} dw + \int_0^w c e^{-dw} dw\right)\left\{\left(1-\alpha(t)\right)\int_0^w r_1(\tau)d\tau + \alpha(t)\int_0^w r_2(\tau)d\tau\right\} \tag{11}$$

By considering the production costs, inventory holding costs, shortage costs, restoration costs and warranty costs, one can have the following formulation for total cost per cycle

TCPC (t, T1) = PC + HC + SC + RC + WC

$$= \left(K + c\,pt\right) + \frac{hp(p-d)(t-T_1)^2}{2d} + \frac{sp(p-d)T_1^2}{2d} + r\left(1-e^{-ut}\right)$$
$$+ \frac{pt}{w}\left(\int_0^w a e^{bw} dw + \int_0^w c e^{-dw} dw\right)\left\{\left(1-\alpha(t)\right)\int_0^w r_1(\tau)d\tau + \alpha(t)\int_0^w r_2(\tau)d\tau\right\} \tag{12}$$

Therefore the objective function can be expressed as follows

$$\text{TCPI}(t) = \frac{K}{pt} + c + \frac{hs(p-d)t}{2d(h+s)}$$
$$+ \frac{r\left(1-e^{-\lambda t}\right)}{pt} + \frac{1}{w}\left(\int_0^w a e^{bw} dw + \int_0^w c e^{-dw} dw\right)\left\{\left(1-\alpha(t)\right)\int_0^w r_1(\tau)d\tau + \alpha(t)\int_0^w r_2(\tau)d\tau\right\} \tag{13}$$

The problem is then to find an optimal production run length t* that leads to the minimum of TCPI (t). The following observations should be mentioned:

1. If s approaches infinity, equation (13) reduces to the expected total relevant cost per item of a no-shortage-allowed case.
2. When r = 0, equation (13) reduces to the case that the restoration cost is not considered.
3. When $\theta_1 = 0$, $\upsilon = 1$ and w = 0, equation (13) reduces to the case that items produced are conforming or are of perfect quality as the production process is in the in-control state. In the case of the out-of-control state, it produces θ_2 fixed percentage of nonconforming items.
4. Combining the arguments of (1) and 3, equation (13) will reduce to equation (1) in Rosenblatt and Lee [3].
5. When $\upsilon = 1$ and w = 0, equation (13) reduces to the case where the production process produces a fixed percentage of nonconforming items θ_1 (or θ_2) in the in-control (or out-of-control) state. It reduces to equation (11) in Hou [4].
6. When $\upsilon = 1$, equation (13) reduces to equation (19) in Chen and Lo [2]. Thus, the proposed model in this paper is an extension of Chen and Lo [2]. In addition, the warranty period is assumed to be a decision variable.

5. Numerical examples

Suppose that the life time distributions of both conforming and nonconforming items are Weibull with hazard rate functions $r_1(\tau) = \lambda_1\beta_1\tau^{\beta_1 - 1}$ and $r_2(\tau) = \lambda_2\beta_2\tau^{\beta_2 - 1}$, respectively. Assume that the shape parameters are β1 = β2 = 2 and the scale parameters are $\lambda_1 = 1/36$ and $\lambda_2 = 1/12$. The remaining model parameters are K = 100, p = 600, d = 400, h = 0.1, s = 0.5, c = 10, a = 0.0125, b = 0.0125, c = 0.0125, d = 0.0125, r = 200, θ1 = 0.15, θ2 = 0.85, υ = 1.645, λ = 0.1.

For the above mentioned deteriorating production system, the optimal production run length under various warranty periods and the corresponding expected total cost per item are summarized in Table 2.

Table 2: Optimal production run lengths under various warranty periods

	w	0	4.875	8.451	12.852	16.837	22.834	26.439	30.812	35.574	40.542
Optimal	t*	2.9266	2.8761	2.7842	2.6321	2.4787	2.2499	2.1218	1.9792	1.8401	1.7116
	TCPI (t*)	10.1468	10.1491	10.1536	10.1621	10.1724	10.1921	10.2061	10.2251	10.2484	10.2756

From Table 2, it is clear that t* decreases and TCPI (t*) increases as w increases.

6. Conclusion

In this paper, we study an imperfect production system with allowable shortages for products sold with free minimal repair warranty. Here we assume that the elapsed time until the production process shift is arbitrarily distributed and time dependent. We develop the mathematical model which is a generalization of Rosenblatt and Lee [3], Chen and Lo [2] and Hou [4]. We show that there exits a unique optimal production run length t* to minimize the expected total relevant cost per unit time. Further more, the effects of the model parameters on the optimal production run length and on the optimal expected total cost per unit time are investigated through a numerical example. According to the results, the expected total cost per unit time is sensitive to the shift rate λ, shortage cost s, restoration cost r, failure distribution shape parameter ν and percent of defective items during in-control, θ_1 and during out of control, θ_2 respectively.

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References
1. Johnson, L.A., and Montgomery, D.C., 1980, Operations Research in Production Planning Scheduling and Inventory Control, McGraw-Hill, New York.
2. Chen, C.K., and Lo, C.C., 2006, “Optimal Production Run Length for Products Sold with Warranty in an Imperfect Production System with Allowable Shortages”, Mathematical and Computer Modeling, 44(3-4), 319-331.
3. Rosenblatt, M.J., and Lee, H.L., 1986, “Economic Production Cycles with Imperfect Production Processes”, IIE Transactions, 18(1), 48–55.
4. Hou, K.L., 2005 “Optimal Production Run Length for Deteriorating Production System with a Two-state Continuous-time Markovian Processes Under Allowable Shortages’, Journal of the Operational Research Society, 56(3), 346-350.