Where $L(m_1, m_2, m_3, m_4, n_1, n_2)$ is defined by (7).

3 Solution of the model

Instead of solving the problem (9) the function *L(m,n)* can be minimized subject to $m_j \geq 1, n_i \geq 1$, i.e., the following two-dimensional nonlinear integer optimization problem is relevant:

$$\min_{(m,n)} L(m,n) = \min_{(m,n)} (P + \sum_{j=1}^{l} R_j m_j + \sum_{i=1}^{k} S_i n_i) \cdot (h_1 + \sum_{j=1}^{l} \frac{h_2^j}{m_j} + \sum_{i=1}^{k} \frac{h_3^i}{n_i}),$$
$$m = (m_1, m_2, \ldots, m_l),\ n = (n_1, n_2, \ldots, n_k) \quad (10)$$
$$m_j, n_i \in \{1, 2, \ldots\}$$

For the solution of the problem (10), consider the following two-dimensional nonlinear integer optimization problem:

$$\min_{(x_1, x_2, \ldots, x_n)} K(x_1, x_2, \ldots, x_n) = \min_{(x_1, x_2, \ldots, x_n)} (b_0 + \sum_{i=1}^{i} b_i x_i) \cdot (a_0 + \sum_{i=1}^{n} \frac{a_i}{x_i}), \quad (11)$$
$$x_i \in \{1, 2, \ldots\},\ \mathrm{i} = 1, 2, \ldots n.$$

First, let us consider the following continuous auxiliary problem:

$$\min_{(x_1, x_2, \ldots, x_n)} K(x_1, x_2, \ldots, x_n) = \min_{(x_1, x_2, \ldots, x_n)} (b_0 + \sum_{i=1}^{i} b_i x_i) \cdot (a_0 + \sum_{i=1}^{n} \frac{a_i}{x_i}), \quad (12)$$
$$x_i \geq 1, i = 1, 2, \ldots, n.$$

By analysing the first partial derivatives, we can prove the following lemma:

Lemma 1. *If* $x_i > 0, i = 1, 2, \ldots, n$, *there are n curves of local minima (12) with respect to* x_j:

$$X_j(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) = \sqrt{\frac{a_j(b_0 + \sum_{i=1, i\neq j}^{n} b_i x_i)}{b_j(a_0 + \sum_{i=1, i\neq j}^{n} \frac{a_i}{x_i})}}, \quad (13)$$

and the point of the local minimum

$$x_j^* = \sqrt{\frac{a_j b_0}{a_0 b_j}},\ i = 1, 2, \ldots, n. \quad (14)$$

Let us denote the radicands of the expressions (14) by

$$A_i = \frac{a_i b_0}{a_0 b_i},\ i = 1, 2, \ldots, n. \quad (15)$$

Without loss of generality, it is supposed that $A_1 < A_2 < \ldots < A_n$.

We denote:

$$B_i(j) = \frac{a_i(b_0 + \sum_{k=1}^{j} b_k)}{b_i(a_0 + \sum_{k=1}^{j} a_k)}, \ i = 1, 2, \ldots, n. \quad (16)$$

Then the optimal solution for the continuous problem (15) is provided by the following theorem.

Theorem 1. *The optimal solution to the problem (12) has the following structure depending on the value of the parameters* $A_i, B_i(j)$:

1. If $A_i \geq 1, i = 1, 2, \ldots, n$, then $x_i = \sqrt{A_i}, \ i = 1, 2, \ldots, n.$
2. If $A_1 < 1$, then consider $B_2(1), B_3(2), \ldots\ldots, B_j(\text{j}-1), \ldots, B_n(n-1)$; if $B_j(\text{j}-1) < 1$ and $B_{j+1}(\text{j}) \geq 1$ then $x_i = 1, \ i = 1, \ldots, j, \ x_i = \sqrt{B_i(j)}, \ i = j+1, \ldots, n.$
3. If $B_n(n-1) < 1$, then $x_i = 1, \ i = 1, \ldots, n.$

This theorem gives the solution for problem (11) for any initial parameters. Also theorem can be used for solution more complicated models. The next section contains numerical illustrations.

4 Numerical examples

Example 1

This section presents numerical example to illustrate the behavior of the model.

Let the parameters of the model be fixed:

$$d = 10000, H_1 = 15, H_2 = 12, h_1 = 10, h_2 = 7,$$
$$R_1 = 30, R_2 = 35, R_3 = 40, R_4 = 45, S_1 = 50, S_2 = 60, P_1 + P_2 = 100,$$
$$\alpha_2 = 0,5, \beta_2 = 0,5, q_1 = 0,2, q_2 = 0,3.$$

We consider different values of parameter $\alpha_1 \in [0,05, 0,75]$ with the step 0,05 and the function $L(\cdot)$ (7) as the the function of α_1. If α_1 changes in $[0,05, 0,75]$, then $\beta_1 = 1 - \alpha_1$ changes in $[0,25, 0,95]$ and $\beta_1 - q_1$ changes in $[0,05, 0,75]$. So the interval $[0,05, 0,75]$ covers all the range of possible values of α_1 with the step 0,05. On the Fig. 4 the total average cost $L(\alpha_1)$ depending on depending on the disposal rate of part 1 α_1 is represented. It is obvious from the Figure 4, that the more disposal rate α_1, the more total average cost $L(\alpha_1)$. It is logic consequence of the fact that the more percentage of final product is accomplished through manufacturing, which is more expensive alternative of remanufacturing. The table 1 represents values of $L(\alpha_1)$ at different α_1 with the step 0,05, optimal lot numbers $(m_1^*, m_2^*, m_3^*, m_4^*, n_1^*, n_2^*)$, and optimal lot sizes $(Q_1^m, Q_2^m, Q_3^m, Q_4^m, Q_1^n, Q_2^n)$
.

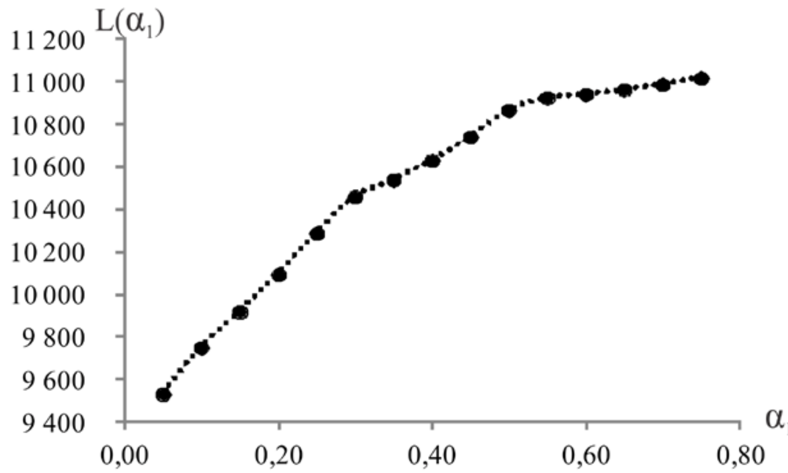

Figure 4: Total average total cost $L(\alpha_1)$ depending on the disposal rate of part 1 α_1

α_1	$L(\alpha_1)$	$(m_1^*, m_2^*, m_3^*, m_4^*, n_1^*, n_2^*)$	$(Q_1^m, Q_2^m, Q_3^m, Q_4^m, Q_1^n, Q_2^n)$	T
0,05	9 531	(3,1,3,2,1,2)	(317,254,85,190,63,317)	0,127
0,1	9 750	(2,1,3,2,1,2)	(413,236,79,177,118,295)	0,118
0,15	9 918	(2,1,2,2,1,2)	(351,216,108,162,162,270)	0,108
0,2	10 095	(2,1,2,2,1,2)	(318,212,106,159,212,265)	0,106
0,25	10 288	(2,1,2,2,1,2)	(286,208,104,156,260,260)	0,104
0,3	10 461	(1,1,1,1,1,1)	(344,138,138,206,206,344)	0,069
0,35	10 538	(1,1,1,1,1,1)	(307,137,137,205,239,342)	0,068
0,4	10 632	(1,1,1,1,1,1)	(271,135,135,203,271,339)	0,068
0,45	10 741	(1,1,1,1,1,1)	(235,134,134,201,302,335)	0,067
0,5	10 866	(1,1,1,1,1,1)	(199,133,133,199,331,331)	0,066
0,55	10 925	(1,1,1,1,2,1)	(188,150,150,225,206,375)	0,075
0,6	10 941	(1,1,1,1,2,1)	(150,150,150,225,225,375)	0,075
0,65	10 962	(1,1,1,1,2,1)	(112,150,150,224,243,374)	0,075
0,7	10 988	(1,1,1,1,2,1)	(75,149,149,224,261,373)	0,075
0,75	11 018	(1,1,1,1,2,1)	(37,149,149,223,279,372)	0,074

Table 1: The results of example 1.

Example 2

Let the parameters of the model be fixed:

$$d = 10000, H_1 = 20, H_2 = 20, h_1 = 7, h_2 = 19,$$
$$R_1 = 30, R_2 = 35, R_3 = 40, R_4 = 100, S_1 = 40, S_2 = 60, P_1 + P_2 = 100,$$
$$\alpha_1 = 0,5, \beta_1 = 0,5, \alpha_2 = 0,1, \beta_2 = 0,9, q_2 = 0,8.$$

We consider different values of parameter $q_1 \in [0,02, 0,48]$ and the function $L(\cdot)$ (7) as the the function of q_1. The question is, how the change of the percentage of remanufactured items changes the total cost. On the Fig. 5 the total average cost $L(q_1)$ depending on the percentage of remanufactured items q_1 is represented. The table 2 represents values of $L(q_1)$ at different q_1 with the step 0,02, optimal lot numbers $(m_1^*, m_2^*, m_3^*, m_4^*, n_1^*, n_2^*)$, and optimal lot sizes $(Q_1^m, Q_2^m, Q_3^m, Q_4^m, Q_1^n, Q_2^n)$.

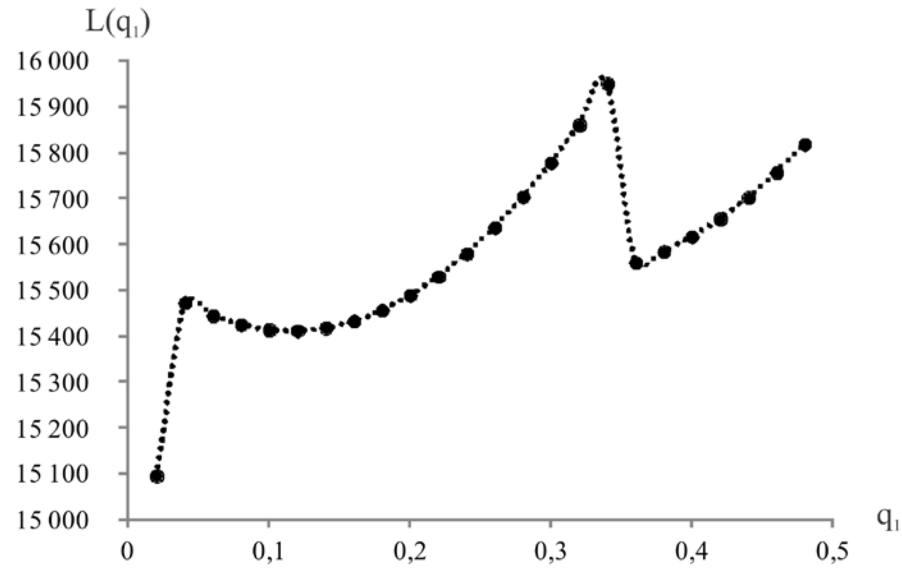

Figure 5: Total average cost $L(q_1)$ depending on the percentage of remanufactured items q_1

If q_1 changes in $[0,02, 0,48]$, then $\beta_1 - q_1$ also changes in $[0,02, 0,48]$. So the interval $[0,02, 0,48]$ covers all the range of possible values of q_1. We can see from the figure 5, that the dependency is rather complex. For instances, the "fall" at $q_1 = 0,36$ can be observed: the optimal policy (1,1,1,3,2,1), which was invariable from $q_1 = 0,04$ to 0,36, was changed to (1,1,3,4,3,1). The total cost increased at the values of q_1 from 0,04 to 0,36. However the policy (1,1,3,4,3,1) remained invariable from $q_1 = 0,36$ to 0,48 and the total cost also increased. The minimum value of total cost 15 094 was reached at $q_1 = 0,02$ under the policy (3,1,1,4,3,1).

q_1	$L(q_1)$	$(m_1^*, m_2^*, m_3^*, m_4^*, n_1^*, n_2^*)$	$(Q_1^m, Q_2^m, Q_3^m, Q_4^m, Q_1^n, Q_2^n)$	T
0,02	**15 094**	**(3,1,1,4,3,1)**	**(179,112,22,224,187,112)**	**0,112**
0,04	**15 471**	**(1,1,1,3,2,1)**	**(384,83,33,222,208,83)**	**0,083**
0,06	15 443	(1,1,1,3,2,1)	(368,84,50,223,209,84)	0,084
0,08	15 423	(1,1,1,3,2,1)	(351,84,67,223,209,84)	0,084
0,1	15 412	(1,1,1,3,2,1)	(335,84,84,223,209,84)	0,084
0,12	**15 410**	**(1,1,1,3,2,1)**	**(318,84,100,223,209,84)**	**0,084**
0,14	15 416	(1,1,1,3,2,1)	(301,84,117,223,209,84)	0,084
0,16	15 431	(1,1,1,3,2,1)	(284,84,134,223,209,84)	0,084
0,18	15 455	(1,1,1,3,2,1)	(267,83,150,223,209,83)	0,083
0,2	15 487	(1,1,1,3,2,1)	(250,83,167,222,208,83)	0,083
0,22	15 528	(1,1,1,3,2,1)	(233,83,183,222,208,83)	0,083
0,24	15 578	(1,1,1,3,2,1)	(215,83,199,221,207,83)	0,083
0,26	15 635	(1,1,1,3,2,1)	(198,83,215,220,206,83)	0,083
0,28	15 702	(1,1,1,3,2,1)	(181,82,230,219,205,82)	0,082
0,3	15 776	(1,1,1,3,2,1)	(164,82,245,218,204,82)	0,082
0,32	15 859	(1,1,1,3,2,1)	(146,81,260,217,203,81)	0,081
0,34	**15 949**	**(1,1,1,3,2,1)**	**(129,81,275,216,202,81)**	**0,081**
0,36	**15 559**	**(1,1,3,4,3,1)**	**(156,111,133,222,185,111)**	**0,111**
0,38	15 583	(1,1,3,4,3,1)	(133,111,141,222,185,111)	0,111
0,4	15 615	(1,1,3,4,3,1)	(111,111,148,222,185,111)	0,111
0,42	15 654	(1,1,3,4,3,1)	(88,111,155,221,184,111)	0,111
0,44	15 701	(1,1,3,4,3,1)	(66,110,162,220,184,110)	0,110

q_1	$L(q_1)$	$(m_1^*, m_2^*, m_3^*, m_4^*, n_1^*, n_2^*)$	$(Q_1^m, Q_2^m, Q_3^m, Q_4^m, Q_1^n, Q_2^n)$	T
0,46	15 755	(1,1,3,4,3,1)	(44,110,168,220,183,110)	0,110
0,48	**15 817**	**(1,1,3,4,3,1)**	**(22,109,175,219,182,109)**	**0,109**

Table 2: The results of example 2.

5 Summary and conclusions

Inventory policies for joint remanufacturing and manufacturing have recently received much attention. Most efforts,though, were related to (optimal) policy structures and numerical optimization, rather than closed form expressions for calculating near optimal policy parameters. The focus of this paper is on the latter. We consider a case of a firm that can manufacture new products and recover the value of a used product through remanufacturing with dismantling for components. The cost structure consists of setup costs, switching cost, holding costs and eoq non related costs. Theorem from the section 3 presents simple, closed form formulae for approximating the optimal policy parameters under a cost minimization objective.

The numerical examples were considered, which demonstrated the alteration of the optimal policy under different values of the initial parameter ceteris paribus. It also demostrates that the dependency of total average cost from other parameters of the model can be rather complex, which is the consequence of integer optimal numbers of optimal policy.

References

[1] E. Akcali, S. Cetinkaya. Quantitative models for inventory and production planning in closed-loop supply chains // International Journal of Production Research. 49(8). P.2373–2407, 2011.

[2] L. Amelia, D.A. Wahab, C.H. Che Haron, N. Muhamad, C.H. Azhari. Initiating automotive component reuse in Malaysia // Journal of Cleaner Production. Vol. 17. Issue 17. P. 1572–1579, 2009.

[3] A. Andriolo, D. Battini, R. W. Grubbström, A. Persona, F.o Sgarbossa. A century of evolution from Harris′s basic lot size model: Survey and research agenda // International Journal of Production Economics. Vol. 155. P. 16–38, 2014.

[4] D.-W. Choi, H. Hwang, S.-G. Koh. A generalized ordering and recovery policy for reusable item // European Journal of Operational Research. 182. P. 764–774, 2007.

[5] I. Dobos, K. Richter. The integer EOQ repair and waste disposal model—further analysis // Central European Journal of Operations Research. Vol. 8. No. 2. P. 173–194, 2000.

[6] I. Dobos, K. Richter. A production/recycling model with stationary demand and return rates // Central European Journal of Operations Research. 11. P. 35–46, 2003.

[7] I. Dobos, K. Richter. An extended production/recycling model with stationary demand and return rates // International Journal of Production Economics. Vol. 90. No. 3. P. 311–323, 2004.

[8] I. Dobos, K. Richter. A production/recycling model with quality consideration // International Journal of Production Economics. Vol. 104. No. 2. P. 571–579, 2006.

[9] M. E. Ferguson, M. Fleischmann, G.C. Souza, G. A profit-maximizing approach to disposition decisions for product returns // Decision Sciences. 42(3). P.

773–798, 2011.

[10] P. Ferrão, J. Amaral. Design for recycling in the automobile industry: new approaches and new tools // J. Eng. Des. 17 (5). P. 447–462, 2007.

[11] K. Govindan, K. M. Shankar, D. Kannan. Application of fuzzy analytic network process for barrier evaluation in automotive components remanufacturing towards cleaner production – a study in an Indian scenario // Journal of Cleaner Production, 2015.

[12] Sh. Guo, G. Aydin , G. C. Souza. Dismantle or remanufacture? // European Journal of Operational Research. 233. P. 580–583, 2014.

[13] P. Hasanov, M.Y. Jaber, S. Zolfaghari. Production, remanufacturing and waste disposal models for the case s of pure and partial backordering // Applied Mathematical Modelling. 36. P. 5249 – 5261, 2012.

[14] M. Y. Jaber,A. M. A. El . The production, remanufacture and waste disposal model with lost sales // International Journal of Production Economics. Vol. 120. No. 1. P. 115–124, 2009.

[15] S.-G. Koh, H. Hwang, K.-I. Sohn, C.-S. Ko // An optimal ordering and recovery policy for reusable items // Computers and Industrial Engineering. Vol. 43. No. 1-2, P. 59–73, 2002.

[16] I. Konstantaras, S. Papachristos. Lot-sizing for a single-product recovery system with backordering // International Journal of Production Research. 44(10), P. 2031–2045, 2006.

[17] I. Konstantaras, S. Papachristos. A note on: Developing an exact solution for an inventory system with product recovery // International Journal of Production Economics. 111. P. 707–712, 2008.

[18] I. Konstantaras, K. Scouri. Lot Sizing for a single product recovery system with variable setup numbers // European Journal of Operational Research. 203. P. 326–335, 2010.

[19] I. Konstantaras, K. Scouri, M.Y. Jaber. Lot sizing for a recoverable product with inspection and sorting // Computer & Industrial Engineering. 58. P. 452–462, 2010.

[20] N. Liu, Y. Kim, H. Hwang., 2009. An optimal operating policy for the production system with rework // Computers and Industrial Engineering. 56. P. 874–887, 2009.

[21] M.C. Mabini, L.M. Pintston, L. F. Gelders. EOQ type formulations for controlling repairable inventories // Int. Journal of Production Economics. 28. P. 21–33, 1992.

[22] S. Nahmias, H. Rivera. A deterministic model for a repairable item inventory system with a finite repair rate // International Journal of Production Research. 17 (3). P. 215–221, 1979.

[23] K. Richter. The EOQ repair and waste disposal model with variable setup numbers // European Journal of Operational Research. 96. P. 313 –324, 1996.

[24] K. Richter. The extended EOQ repair and waste disposal model // International Journal of Production Economics. Vol. 45. No. 1–3. P. 443–447, 1996.

[25] K. Richter. Pure and mixed strategies for the EOQ repair and waste disposal problem // OR Spectrum. Vol. 19. No. 2. P. 123–129, 1997.

[26] K. Richter, I. Dobos. Analysis of the EOQ repair and waste disposal problem with integer setup numbers // International Journal of Production Economics. Vol. 59. No. 1–3. P. 463–467, 1999.

[27] D. S. Rogers, R.S. Tibben-Lembke. Going Backwards: Reverse Logistics Trends and Practices. university of Nevada, Reno, Center for Logistics Management, 1998.

[28] A. M. A. El Saadany, M. Y. Jaber. The EOQ repair and waste disposal model with switching costs // Computers & Industrial Engineering. Vol. 55. Vo. 1. P. 219–233, 2008.

[29] A.M.A. El Saadany, M.Y. Jaber. A production, repair and waste disposal inventory model when returns are subject to quality and price considerations // Computers & Industrial Engineering. 58 (3). P. 352–362, 2010.

[30] A. M.A. El Saadany , M. Y. Jaber, M. Bonney. How many times to remanufacture? // Int. J. Production Economics. 143. P. 598–604, 2013.

[31] D.A. Schrady. A deterministic inventory model for reparable items // Naval Research Logistics Quarterly. 14 (3). P. 391–398, 1967.

[32] G.C. Souza. Closed-loop supply chains: A critical review, and future research // Decision Sciences. 44(1). P. 7–38, 2013.

[33] R. H. Teunter. Economic ordering quantities for recoverable item inventory systems // Naval Research Logistics. Vol. 48. No. 6. P. 484–495, 2001.

[34] J. Tian, M. Chen. Sustainable design for automotive products: Dismantling and recycling of end-of-life vehicles // Waste Management. Vol. 34. Issue 2. P. 458–467, 2014.

[35] X. Xia, K. Govindan, Q. Zhu. Analyzing internal barriers for automotive components remanufacturers in China using grey-DEMATEL approach // Journal of Cleaner Production. Vol. 87. P. 811–825, 2015.