Optimal Promotional Effort and Pricing Policies for a Multiple Item Innovation Diffusion Model involving Fuzzy Parameters

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Abstract

With immense competition, coupled with short and unpredictable shelf life of products in some cases, companies rely upon diversifying their product range to disperse risk associated with failure of one product. Since pricing is also dynamic, varying with time, there is also a need for the companies to formulate appropriate pricing policies to maximize profits as against expenses incurred in promotion of multiple products. In this paper we device and analyze policies for promoting sales of multiple products in the market through optimal control theory problems. The main aim is to maximize profits while considering multiple product range, demand requirements and optimum pricing policy for an available budget for promotion. To be as near as possible to a real-life situation, the total available budget is taken to be imprecise. Further, optimal control model with fuzzy parameter is converted into crisp form by using both necessity and possibility constraints. The solution is derived using Pontryagin Maximum principle and the entire technique is illustrated by giving a numerical example.

Keywords
Fuzzy parameter, Optimal control problem, Multiple Item, Time dependent price

1. Introduction

Never keep all your eggs in the same basket. This is a famous saying that gives importance to diversifying risk and teaches not to put all stakes at one place. In today's competitive times, a company does not rely on a single product for its business. To survive, it has become necessary that a company diversify its product range such that failure of one product does not fatally impact the company and the risk gets spread out over the entire product range. The shelf life of products is reducing with the advent of newer technologies and improvements. A product is replaced within a few months with a new product having significant improvements or advancement in technology. The new product could be from a competitor or the same company. Be that as it may be, with shorter shelf-lives, every manufacturer these days is manufacturing multiple products and allocating budgets for advertisement separately for each product with an aim to maximize profits.

In the field of optimization, the role of optimal control theory has been very vital in dealing with problems that may have some dynamic parameters. Bass (1969) played a major role in providing with some basic models representing diffusion of innovation in markets, which, with some modifications, are relevant even till today. Numerous authors have carried out study of various factors on the diffusion of innovation of a new product. Effect of factors like price and publicity of the product has been studied by Robinson and Lakhani (1975) and Horsky and Simmon (1983), respectively.

Various advertising models which employ the technique of optimal control theory have studied by Little and Lodish (1969), Grosset and Viscolani (2005). Jha et al. (2009) in their model have partitioned the market into various homogenous clusters in order to provide better advertising strategies.

The above mentioned models had assumed all parameters to be defined crisply. But in a lot of situations there may be some uncertainty associated with the parameters used. To deal with such impreciseness, Zadeh (1965) had given the basic idea of fuzzy sets. His work in the field of fuzzy numbers and intervals provided means to tackle such vagueness.

Application of optimal control theory techniques for analyzing a model with fuzzy objective function was illustrated by Filev & Angelov (1992). Later, Maity and Maiti (2005) have considered fuzzy production and inventory model for products which deteriorate with time. The use of optimal control theory in the area of portfolio optimization has been displayed by Zhu (2009). The model considered had fuzziness in the returns of assets.

In the production models with fuzzy production costs Roul et al. (2019) have used the technique of optimal control theory aiming to minimize the production costs. Campos et al. (2020) have used fuzzy intervals to represent the imprecise parameters in a similar problem.

Various production and inventory models which have employed the techniques of optimal control theory in the case when firms produce more than one product have also been given in the past. Parameters of purchasing cost, the amount invested and the storage capacity considered to be fuzzy in nature. The problems analyzed by Maity & Maiti (2007) and Mandal et al. (2010) study the effect of production of defective products on the entire process. They have further assumed parameters like storage and production costs to be fuzzy in nature. Mandal et. al. (2011) have studied another production problem which is associated with multiple products. The preparation time is taken as a variable and the space constraint is taken to be fuzzy in nature and an optimal control model is formulated for the problem and solved using Genetic Algorithm. An optimization problem for a multi-item inventory model is considered by Maiti (2020). Parameters of purchasing cost, the amount invested and the storage capacity considered to be fuzzy in nature.

A product's price does not remain static but varies with time depending upon various factors. Conventionally, price of a product has been dependent on demand and supply but with numerous substitutes available, demand for a particular product now also depends on the price and as more and more substitutes and options become available to a consumer, prices are varied over time to increase demand as against other choices. This paper aims to maximize profits for companies producing multiple products and allocates separate imprecise budget for advertisement of each product assuming that the price of each product varies with time. The maximization of profit is proposed through optimum control theory with fuzzy parameter being converted into crisp form by using both necessity and possibility constraints. This process of defuzzification is based on methods given by Dubois and Prade (1988,1997), and Liu and Iwamura, (1998). The crisp model aiming to maximize profits is then solved by using Pontryagin's maximum principle.

2. Basic Concepts

In this section some basic terms related to fuzzy sets will be introduced.

2.1 Triangular Fuzzy Number

A fuzzy set $\tilde{X}$ in R is said to be a triangular fuzzy number if its membership function is given by:

$$\mu_{\tilde{X}}(x) = \begin{cases} 0, & x < a_1 \\ \frac{x-a_1}{a_2-a_1}, & a_1 \le x \le a_2 \\ \frac{a_3-x}{a_3-a_2}, & a_2 < x \le a_3 \\ 0, & x > a_3 \end{cases}$$

The number $\tilde{X}$ is denoted by the triplet (a_1, a_2, a_3) and it's membership function is in the form of a triangle, as depicted by figure 1.

Figure 1: Membership function for the triangular fuzzy number

2.2 Operations on Fuzzy numbers

Using Zadeh's Extension principle given in Zadeh (1965), the fuzzy number $\tilde{X} * \tilde{Y}$ is given by the membership function:

$$\mu_{\tilde{X}*\tilde{Y}}(z) = \sup_{z=x*y;z\in R} min\left(\mu_{\tilde{X}}(x), \mu_{\tilde{Y}}(y)\right)$$

For the proposed model we would be referring to triangular fuzzy numbers given by $\tilde{\mathbf{X}}$ (a_1, a_2, a_3) and $\tilde{Y}$ given by (b_1, b_2, b_3). It can be shown that for these fuzzy numbers $\tilde{\boldsymbol{X}}$ (+) $\tilde{Y}$, $\tilde{\boldsymbol{X}}$ (-) $\tilde{Y}$, -$\tilde{\boldsymbol{X}}$ and k$\tilde{\boldsymbol{X}}$ are also triangular fuzzy numbers with $\tilde{\boldsymbol{X}}$ (+) $\tilde{Y}$ given by ($a_1 + b_1$, $a_2 + b_2$, $a_3 + b_3$), -$\tilde{Y}$ given by ($-b_3$, $-b_2$, $-b_1$), K$\tilde{\boldsymbol{X}}$ given by (ka_1, ka_2, ka_3), and $\tilde{\boldsymbol{X}}$ (-) $\tilde{Y}$ given by ($a_1 - b_3$, $a_2 - b_2$, $a_3 - b_1$).

2.3 Possibility and Necessity on Fuzzy numbers

Let $\tilde{X}$ and $\tilde{Y}$ be two fuzzy numbers in R with membership functions $\mu_{\tilde{X}}(x)$ and $\mu_{\tilde{Y}}(y)$. As given by Zadeh (1999), Dubois and Prade, (1988,1997) and Liu and Iwamura, (1998), the Possibility and Necessity of certain events in the fuzzy environment are given as follows:
Pos ($\tilde{X} \leq \tilde{Y}$) = $sup\{min(\mu_{\tilde{X}}(x)$, $\mu_{\tilde{Y}}(y))$: $x, y \in$ R, $x \leq y\}$
Pos($\tilde{X} < \tilde{Y}$)= $sup\{min(\mu_{\tilde{X}}(x)$, $\mu_{\tilde{Y}}(y))$: $x, y \in$ R, $x < y\}$
and Pos($\tilde{X} = \tilde{Y}$)= $sup\{ \mu_{\tilde{X}}(x)$, $\mu_{\tilde{Y}}(x))$: $x \in$ R$\}$
If $\tilde{X}$=c is crisp real number, then the above possibilities become:
Pos(c $\leq \tilde{Y}$)= $sup\{min(\mu_{\tilde{Y}}(y))$: $x, y \in$ R, $c \leq y\}$
Pos(c $< \tilde{Y}$)= $sup\{min(\mu_{\tilde{Y}}(y))$: $x, y \in$ R, $c < y\}$
and Pos(c = $\tilde{Y}$)= $\mu_{\tilde{Y}}(c)$
Further, the Necessity is defined as follows:
Nes ($\tilde{X} * \tilde{Y}$)= inf $\{max(\mu_{\tilde{X}}(x)$, $\mu_{\tilde{Y}}(y))$: $x, y \in$ R, $x * y\}$, where * represents any one of the relational operators $<, \leq, =, >, \geq$.
The necessary and possible events thus share a dual relationship given by
Nes ($\tilde{X} * \tilde{Y}$)= 1- Pos ($\overline{\tilde{X} * \tilde{Y}}$)
In particular for two triangular fuzzy numbers $\tilde{X}$ = (X_1, X_2, X_3) and $\tilde{Y}$ = (Y_1, Y_2, Y_3) the value of Pos ($\tilde{X} \leq \tilde{Y}$) is given by the following cases:

Case 1: When $X_2 \leq Y_2$, Pos ($\tilde{X} \leq \tilde{Y}$) =1

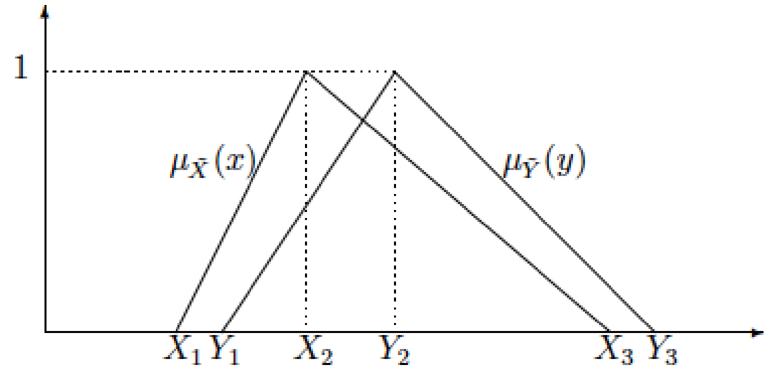

Figure 2: Membership function for two triangular fuzzy numbers with $X_2 \leq Y_2$

Case 2: When $X_2 > Y_2$ and $Y_3 > X_1$, Pos($\tilde{X} \leq \tilde{Y}$) = δ, where $\delta = \frac{Y_3 - X_1}{X_2 - X_1 + Y_3 - Y_2}$

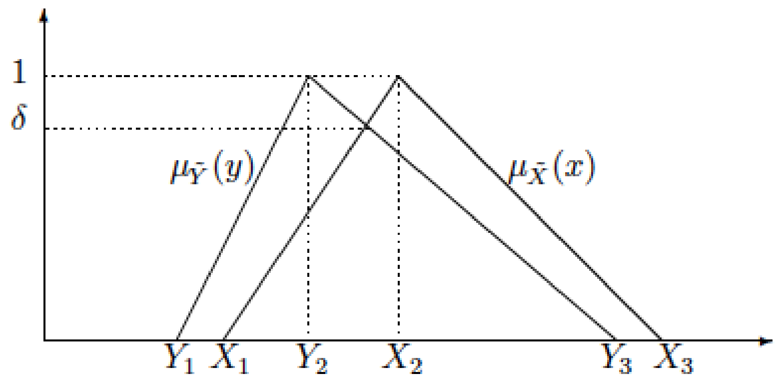

Figure 3: Membership function for two triangular fuzzy numbers with $X_2 > Y_2$ and $Y_3 > X_1$

Case 3: When $X_1 \geq Y_3$, Pos($\tilde{X} \leq \tilde{Y}$) = 0

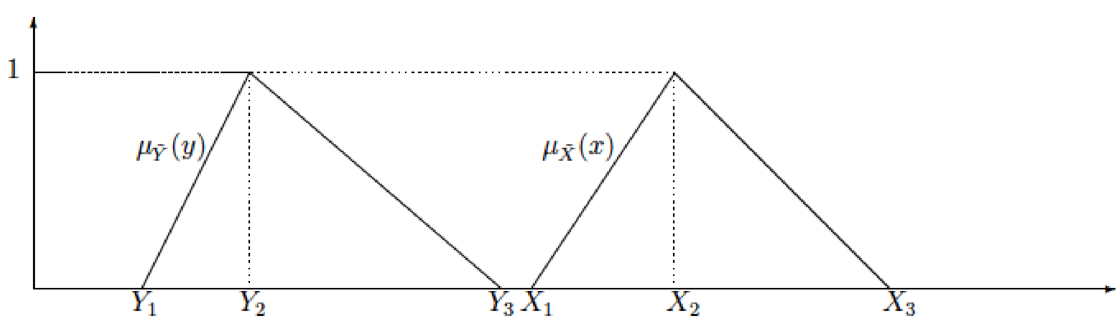

Figure 4: Membership function for two triangular fuzzy numbers with $X_1 \geq Y_3$

Hence Pos($\tilde{X} \leq \tilde{Y}$) can be given as:

$$\text{Pos}(\tilde{X} \leq \tilde{Y}) = \begin{cases} 1 & , X_2 \leq Y_2 \\ \delta = \frac{Y_3 - X_1}{X_2 - X_1 + Y_3 - Y_2} & , X_2 > Y_2 \text{ and } Y_3 > X_1 \\ 0 & , X_1 \geq Y_3 \end{cases} \quad (1)$$

3. Programming Problem in fuzzy environment

Let us consider the following programming problem with fuzzy parameters:
Maximize J
subject to $g \leq \tilde{Y}$ (FP)
where J is the objective function and g is the constraint involving the fuzzy parameter $\tilde{Y}$.
To convert the constraints to their equivalent crisp versions, similar to the concepts given by Liu and Iwamura (1998), we can write the problem (FP) problem under necessity and possibility constraints as
Maximize J
subject to $Nes\{g \leq \tilde{Y}\} \geq \eta_1$ and/or $\mathrm{Pos}\{g \leq \tilde{Y}\} \geq \eta_2$
where η_1 and η_2 are predetermined confidence levels for the fuzzy constraint.

4. Model Notations

To formulate the model, the following notations have been used in the paper:

N : the total number of products which is a discrete variable
$\overline{X}_i$: the total number of potential customers for the i^{th} product
$x_i(t)$: the sales for the i^{th} product at time t
$u_i(t)$: the promotional effort rate for the i^{th} product at time t
p_i : the coefficients of external influence for the i^{th} product
q_i : the coefficients of internal influence for the i^{th} product
$\phi_i\big(u_i(t)\big) = \frac{\epsilon_i}{2} u_i^2(t)$: the promotional effort cost for the i^{th} product at time t
$P_i(t)$: the sales price per unit for the i^{th} product at time t
d_i : price parameter for the i^{th} product
C_i : the production cost for the i^{th} product
$\widetilde{W}$: fuzzy variable representing the total available budget

5. Model Development

The model considered assumes that the firm produces N products. The firm puts in varied efforts to promote its products in the market and the objective is to maximize its profits. The total available budget for promotion is taken to be fuzzy in nature and the price of the products is taken to be varying with time. The control theory model for the above mentioned problem is formulated as follows:

$$Maximize\, J = \int_0^T \left(\sum_{i=1}^N \left[(P_i(t) - C_i)\dot{x}_i(t) - \frac{\epsilon_i}{2} u_i^2(t)\right]\right) dt \quad (2)$$

subject to

$$\dot{x}_i(t) = \left(p_i + q_i \frac{x_i(t)}{\overline{X}_i}\right)\big(u_i(t)\big)\big(\overline{X}_i - x_i(t)\big)\, e^{-d_i P_i(t)}, \quad (3)$$

$$x_i(0) = x_{i0};\ u_i^l \leq u_i(t) \leq u_i^u\ ;\ P_i^l \leq P_i(t) \leq P_i^u\ \forall\, i = 1 \dots N; \quad (4)$$

$$\int_0^T \left(\sum_{i=1}^N \frac{\epsilon_i}{2} u_i^2(t)\right) dt \leq \widetilde{W} \quad (5)$$

Using properties of definite Integral, we get from equation (5)

$$\sum_{i=1}^N \frac{\epsilon_i}{2} u_i^2(t) \leq \frac{\widetilde{W}}{T} \quad (6)$$

There are two different forms of the fuzzy constraint given by equation (6) depicting two different scenarios.

Scenario 1

$$Nes\{\sum_{i=1}^N \frac{\epsilon_i}{2} u_i^2(t) < \frac{\widetilde{W}}{T}\} \geq \eta_1 \quad (7)$$

This can also be written as

$$Pos\{\sum_{i=1}^N \frac{\epsilon_i}{2} u_i^2(t) \geq \frac{\widetilde{W}}{T}\} \leq 1 - \eta_1$$

Scenario 2

$$Pos\{\sum_{i=1}^N \frac{\epsilon_i}{2} u_i^2(t) \leq \frac{\widetilde{W}}{T}\} \geq \eta_2 \quad (8)$$

5.1 Equivalent crisp representation of the proposed model

Let $\widetilde{W} = (W_1, W_2, W_3)$ be a triangular fuzzy number. Then $\frac{\widetilde{W}}{T} = (W_1/T, W_2/T, W_3/T)$ be given by the triangular fuzzy number $\left(W_1', W_2', W_3'\right)$.

The problem represented by (2)-(6) reduces to the following:

$$Maximize\ J = \int_0^T \left(\sum_{i=1}^N \left[(P_i(t) - C_i)\dot{x}_i(t) - \frac{\epsilon_i}{2} u_i^2(t)\right]\right) dt \tag{9}$$

subject to (3) and (4) for all scenarios and

for Scenario 1

$$\frac{\sum_{i=1}^N \frac{\epsilon_i}{2} u_i^2(t) - W_1'}{W_2' - W_1'} \leq 1 - \eta_1 \tag{10}$$

for Scenario 2

$$\frac{W_3' - \left(\sum_{i=1}^N \frac{\epsilon_i}{2} u_i^2(t)\right)}{W_3' - W_2'} \geq \eta_2 \tag{11}$$

The above crisp problem subject to the constraints is solved by first considering the Hamiltonian given by:

$$\boldsymbol{H = \sum_{i=1}^N \left[(P_i(t) - C_i)\dot{x}_i(t) - \frac{\epsilon_i}{2} u_i^2(t)\right] + \sum_{i=1}^N \mu_i(t)\dot{x}_i(t)} \tag{12}$$

The corresponding Lagrangian is

for Scenario 1

$$\boldsymbol{L = H + \lambda\left[(1 - \eta_1)W_2' + \eta_1 W_1' - \left(\sum_{i=1}^N \frac{\epsilon_i}{2} u_i^2(t)\right)\right]} \tag{13}$$

and for Scenario 2

$$\boldsymbol{L = H + \lambda\left[(1 - \eta_2)W_3' + \eta_2 W_2' - \left(\sum_{i=1}^N \frac{\epsilon_i}{2} u_i^2(t)\right)\right]} \tag{14}$$

In general for Scenario k=1,2, L may be given by

$$\boldsymbol{L = H + \lambda\left[(1 - \eta_k)W_{k+1}' + \eta_k W_k' - \left(\sum_{i=1}^N \frac{\epsilon_i}{2} u_i^2(t)\right)\right]} \tag{15}$$

K-T Conditions for the $\boldsymbol{k^{th}(k = 1, 2)}$ Scenario are

$$\boldsymbol{\lambda\left[(1 - \eta_k)W_{k+1}' + \eta_k W_k' - \left(\sum_{i=1}^N \frac{\epsilon_i}{2} u_i^2(t)\right)\right] = 0} \tag{16}$$

The adjoint functions $\boldsymbol{\mu_i(t)}$ are obtained by using the Maximum Principle as follows:

$$\boldsymbol{\frac{d}{dt}\mu_i(t) = -\frac{\partial L}{\partial x_i(t)}} \tag{17}$$

$$\boldsymbol{\mu_i(T) = 0} \tag{18}$$

$$\boldsymbol{Now, \frac{\partial L}{\partial x_i(t)} = \left(P_i(t) - C_i + \mu_i(t)\right)\frac{\partial \dot{x}_i(t)}{\partial x_i(t)}}$$

$$\boldsymbol{= \left(P_i(t) - C_i + \mu_i(t)\right)\left(u_i(t)\right)\left(q_i - p_i - 2q_i\frac{x_i(t)}{\overline{X_i}}\right)e^{-d_i P_i(t)}} \tag{19}$$

Hence from equation (17), $\boldsymbol{\mu_i(t)}$ satisfies equation (18) and is given by

$$\boldsymbol{\frac{d}{dt}\mu_i(t) = -\left(P_i(t) - C_i + \mu_i(t)\right)\left(u_i(t)\right)\left(q_i - p_i - 2q_i\frac{x_i(t)}{\overline{X_i}}\right)e^{-d_i P_i(t)}} \tag{20}$$

According to the Maximum principle, the Lagrangian is to be maximised with respect to the control variables $\boldsymbol{u_i(t)\ and\ P_i(t)}$, $\forall \boldsymbol{i = 1 \ldots N}$ at every instant of time. Using equation (15) we get:

$$\frac{\partial L}{\partial u_i(t)} = -\epsilon_i u_i(t) + \left(P_i(t) - C_i + \mu_i(t)\right)\left(p_i + q_i \frac{x_i(t)}{\overline{X_\iota}}\right)\left(\overline{X_\iota} - x_i(t)\right)e^{-d_i P_i(t)} - \lambda\epsilon_i u_i(t)$$
(21)

and

$$\frac{\partial^2 L}{\partial u_i^2(t)} = -\epsilon_i(\lambda + 1) \tag{22}$$

Letting $A = \frac{\left(P_i(t)-C_i+\mu_i(t)\right)\left(p_i+q_i\frac{x_i(t)}{\overline{X_\iota}}\right)\left(\overline{X_\iota}-x_i(t)\right)e^{-d_iP_i(t)}}{\epsilon_i(\lambda+1)}$, we can infer the following three cases:

Case 1: If $\frac{\partial L}{\partial u_i(t)} > 0$ then the Lagrangian L is an increasing function of $u_i(t)$ $\boldsymbol{and}$ $u_i^*(t) = \boldsymbol{minimum}\{A, u_i^u\}$

Case 2: If $\frac{\partial L}{\partial u_i(t)} = 0$ then L is maximised for the value of $u_i(t)$ as $u_i^*(t) = A \ (\boldsymbol{for}\ \lambda \geq 0)$

Case 3: If $\frac{\partial L}{\partial u_i(t)} < 0$ then L is a decreasing function of $u_i(t)$ and $u_i^*(t) = \boldsymbol{maximum}\{A, u_i^l\}$

Further using equation (15), we get:

$$\frac{\partial L}{\partial P_i(t)} = \dot{x}_\iota(t)\left(1 - d_i P_i(t) - d_i \mu_i(t)\right)$$

and $\frac{\partial^2 L}{\partial P_i^2(t)} = -d_i\dot{x}_\iota(t)\left(2 - d_iP_i(t) - d_i\mu_i(t)\right)$

We can infer the following three cases:

Case 1: If $\frac{\partial L}{\partial P_i(t)} > 0$ then the Lagrangian L is an increasing function of $P_i(t)$ $\boldsymbol{and}$ $P_i^*(t) = \boldsymbol{minimum}\{\frac{1-d_i\mu_i(t)}{d_i}, P_i^u\}$

Case 2: If $\frac{\partial L}{\partial P_i(t)} = 0$ then L is maximised for the value of $P_i(t)$ as $P_i^*(t) = \frac{1-d_i\mu_i(t)}{d_i}$

Case 3: If $\frac{\partial L}{\partial P_i(t)} < 0$ then L is a decreasing function of $P_i(t)$ and $P_i^*(t) = \boldsymbol{maximum}\{\frac{1-d_i\mu_i(t)}{d_i}, P_i^l\}$

6. Numerical Illustration

A numerical example is presented here to demonstrate the usage of the above proposed model. The equivalent crisp control theory problem given by Eqns (9) - (11) is transformed to a discrete form by using the technique proposed by Rosen (1968). The following discrete formulation of the problem is considered for its numerical application:

$$Maximize\ J = \sum_{k=1}^{T}\left(\sum_{i=1}^{N}\left[(P_i(k) - C_i)(x_i(k+1) - x_i(k)) - \frac{\epsilon_i}{2}u_i^2(k)\right]\right) \tag{25}$$

subject to

$$x_i(k+1) = x_i(k) + \left(p_i + q_i\frac{x_i(k)}{\overline{X_\iota}}\right)\left(u_i(k)\right)\left(\overline{X_\iota} - x_i(k)\right)e^{-d_iP_i(k)}, \tag{26}$$

$$x_i(0) = x_{i0};\ u_i^l \leq u_i(k) \leq u_i^u\ ;\ P_i^l \leq P_i(k) \leq P_i^u\ \forall\ i = 1 \dots N, \forall\ k = 1 \dots T \tag{27}$$

for all scenarios and

for Scenario 1

$$\frac{\sum_{i=1}^{N}\frac{\epsilon_i}{2}u_i^2(k) - W_1'}{W_2' - W_1'} \leq 1 - \eta_1\ \forall\ k = 1 \dots T \tag{28}$$

for Scenario 2

$$\frac{W_3' - \left(\sum_{i=1}^{N}\frac{\epsilon_i}{2}u_i^2(k)\right)}{W_3' - W_2'} \geq \eta_2\ \forall\ k = 1 \dots T \tag{29}$$

The above model is solved by using Lingo11. The value of N which denotes the number of products is taken as 3. The entire time interval is assumed to be divide into 24 time periods of equal duration. The values taken for the different parameters used is tabulated below:

Table 1: Values of parameters

Products	1	2	3
$\overline{X_i}$	10000	25000	30000
p_i	0.00166	0.004161	0.00538
q_i	0.3176	0.380575	0.340395
C_i	340	370	340
ϵ_i	128370	126080	127050
d_i	0.0008	0.0008	0.0008
Initial Sales	1760	8231	8260

The value of T is taken as 24. The range for $u_i(k)$ is taken as $0.1 \leq u_i(k) \leq 1 \; \forall \; i = 1 \dots 3, \forall \; k = 1 \dots 24$. The range for price is taken as $335 \leq P_1(k) \leq 390; 390 \leq P_2(k) \leq 435$ and $380 \leq P_3(k) \leq 395 \; \forall \; k = 1 \dots 24$.
The total available budget which is assumed to be fuzzy in nature, is represented by the triangular fuzzy number and the value of $(W_1^{'}, W_2^{'}, W_3^{'})$ is taken as (22000, 25000, 27500).

6.1 Scenario 1

The value of η_1 is taken as 0.05. Table 2 gives the values of optimal sales of each product over the 24 time periods.

Table 2: The optimal sales for the three products

Time Periods	$x_1(t)$	$x_2(t)$	$x_3(t)$
T1	1760	8231	8260
T2	1835.768	8837.005	8936.896
T3	1914.006	9464.252	9645.115
T4	1994.625	10109.42	10381.75
T5	2077.555	10768.98	11143.57
T6	2162.784	11439.61	11927.36
T7	2250.292	12117.66	12729.44
T8	2340.055	12799.32	13545.71
T9	2432.042	13480.64	14371.7
T10	2526.216	14157.62	15202.7
T11	2622.536	14826.37	16033.81
T12	2720.954	15483.11	16860.1
T13	2821.418	16124.32	17676.7
T14	2923.868	16746.78	18478.91
T15	3028.24	17347.66	19262.33
T16	3134.467	17924.55	20022.94
T17	3242.475	18475.47	20757.18
T18	3352.184	18998.92	21462.01
T19	3463.511	19493.86	22134.95
T20	3576.366	19959.65	22774.1
T21	3690.657	20396.1	23378.12
T22	3806.285	20803.36	23946.22
T23	3923.147	21181.9	24478.12
T24	4041.135	21532.48	24974.01

Table 3 gives the optimal promotional efforts for the three products.

Table 3: Optimal promotional efforts

Time Periods	$u_1(t)$	$u_2(t)$	$u_3(t)$
T1	0.2182508	0.3953327	0.4309395
T2	0.2183297	0.3962932	0.4318996
T3	0.2181114	0.3967535	0.4323265
T4	0.2176988	0.3968897	0.4324126
T5	0.2172714	0.3970243	0.4325071
T6	0.2168314	0.3971559	0.4326103
T7	0.2163812	0.3972833	0.432722
T8	0.2159234	0.3974049	0.4328422
T9	0.2154608	0.3975198	0.4329704
T10	0.2149966	0.3976267	0.4331061
T11	0.2145339	0.3977246	0.4332487
T12	0.2140757	0.3978129	0.4333972
T13	0.2136254	0.3978909	0.4335506
T14	0.2131859	0.3979584	0.4337077
T15	0.21276	0.3980153	0.4338672
T16	0.2123504	0.3980616	0.4340278
T17	0.2119593	0.3980977	0.434188
T18	0.2115887	0.3981242	4.34E-01
T19	0.2112402	3.98E-01	0.434502
T20	0.210915	0.3981509	0.4346532
T21	0.2106139	0.3981528	0.434799
T22	0.2103375	0.3981482	0.4349383
T23	0.2100859	0.3981379	0.4350705
T24	0.209859	0.3981228	0.4351948

The optimal price for the three products is obtained as $P_1(t) = 390$, $P_2(t) = 435$ and $P_3(t) = 395$, over the 24 time periods. The optimal profit obtained is 1354014 units.

6.2 Scenario 2

The value of η_2 is taken as 0.5. Table 4 gives the values of optimal sales of each product over the 24 time periods.

Table 4: The optimal sales for the three products

Time Periods	$x_1(t)$	$x_2(t)$	$x_3(t)$
T1	1760	8231	8260
T2	1836.222	8827.432	8926.606
T3	1914.945	9444.442	9623.592
T4	1996.189	10079.71	10349.15
T5	2079.97	10730.54	11100.95
T6	2166.301	11393.87	11876.17
T7	2255.185	12066.34	12671.47
T8	2346.62	12744.34	13483.06
T9	2440.598	13424.06	14306.74
T10	2537.103	14101.59	15137.94
T11	2636.11	14773.02	15971.87
T12	2737.588	15434.5	16803.59
T13	2841.498	16082.33	17628.1
T14	2947.789	16713.06	18440.51

T15	3056.406	17323.59	19236.1
T16	3167.28	17911.16	20010.49
T17	3280.338	18473.47	20759.69
T18	3395.495	19008.64	21480.2
T19	3512.655	19515.29	22169.1
T20	3631.488	19991.49	22822.65
T21	3751.841	20436.82	23439.18
T22	3873.599	20851.48	24017.86
T23	3996.64	21236	24558.46
T24	4120.837	21591.22	25061.22

Table 5 gives the optimal promotional efforts for the three products.

Table 5: Optimal promotional efforts

Time Periods	$u_1(t)$	$u_2(t)$	$u_3(t)$
T1	0.2195598	0.389088	0.4243887
T2	0.2196401	0.3900043	0.4253065
T3	0.2197232	0.3909659	0.4262790
T4	0.2198091	0.3919711	0.4273057
T5	0.219898	0.3930171	0.4283853
T6	0.2199898	0.3941006	0.4295158
T7	0.2200845	0.3952177	0.4306944
T8	0.2201822	0.3963634	0.4319174
T9	0.2202829	0.3975324	0.4331803
T10	0.2203866	0.3987186	0.4344775
T11	0.2204933	0.3999155	0.4358028
T12	0.2206029	0.4011162	0.4371492
T13	0.2207156	0.4023134	0.4385091
T14	0.2208311	0.4035002	0.4398743
T15	0.2209496	0.4046695	0.4412366
T16	0.221071	0.4058144	0.4425874
T17	0.2211951	0.4069287	0.4439185
T18	0.2213219	0.4080068	0.4452218
T19	0.221023	0.4081828	0.4455595
T20	0.2206725	0.4081943	0.4457246
T21	0.2203492	0.4081973	0.4458834
T22	0.2200536	0.408193	0.4460348
T23	0.219786	0.4081822	0.4461779
T24	0.2195461	0.408166	0.4463119

The optimal price for the three products is obtained as $P_1(t) = 390$, $P_2(t) = 435$ and $P_3(t) = 395$, over the 24 time periods. The optimal profit obtained is 1354361 units.

7. Discussion and Conclusion

Here we see that the total expenditure on promotion in the first scenario is 596216.5531 units whereas in the second scenario it is 609286.2793 units. Thus, we observe that a higher level of expenditure on promotion results in increase in the profit for a fuzzy (variable) budget. The triangular fuzzy number representing the total budget is given by (528000, 600000, 660000), thereby showing that the expense in the first scenario lies in the lower interval (528000, 600000) , whereas the expense in the second scenario lies in the upper interval (600000, 660000). We also note that the increase in promotional effort rates results in the increase in sales.

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