Properties of Fractional Brownian Motions for Modeling Stock Prices

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Abstract

Fractional Brownian motion is a general form of Brownian motion by adding a parameter / index, namely the Hurts index. Modeling stock prices with Brownian motion is common. In this article we will discuss fractional Brown motion for modeling stock prices. Some of these properties are increments which are normally distributed and not mutually independent, self-similarity and long-range dependent.

Keywords:
Fractional Brown motion, Stock price, Hurst index, Properties.

1. Introduction

In the world of investment in stock assets, Value-at-Risk (VaR) gives investors an indication of portfolio risk. A portfolio is a collection of several pieces of share assets owned by an investor (Kalfin *et al*., 2019a; Kalfin *et al*., 2019b). Incremental Value-at-Risk (IVaR) provides an indication of how these risks change when there is a change in portfolio investment (Calvalcante and Assaaf, 2004; Kang, Cheong and Yoon, 2010; Sukono et al., 2019). In practice, it is often related to changes in portfolio risk when investors make changes to new investments. Therefore, a portfolio risk analysis is required when investing (Hasbullah et al., 2020; Sukono et al., 2020; Kalfin et al., 2020). In this case, IVaR is a change in the VaR portfolio that is related to each addition of new investments in the portfolio.

The relationship between IVaR and new investments is very informative (Goddard and Onali, 2012; Murwaningtyas *et al*., 2016). Size IVaR can be used as an aid in making risk control decisions. For example, investors can use IVaR to determine the expected return on investment plans, and to determine investment limits. While Component Value-at-Risk (CVaR) is very useful for identifying high risk sources and also the opposite, and for determining investment constraints, making investment decisions, determining capital requirements, etc.

2. Methods

The methodology of this research is literature study. In this presentation, we propose some important properties of fractional Brownian motion for stock prices modeling. These properties are stationary increments, self-similarity and short and long memory.

3. Results and Discussion

Definition 1
Brownian motion is stochastic processes $(B_t)_{t\geq 0}$ with

a. $B_0 = 0$ almost surely;
b. For every finite sequential time $t_0 < t_1 < \ldots\ldots < t_n$, *increments* $B_{t_1} = B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots., B_{t_n} - B_{t_{n-1}}$ are independent;
c. For $0 \leq s < t$, random variabel $B_t - B_s$ normally distributed with expectation 0 and variance $t - s$.

Definition 2
Brownian motion is centralized Gaussian processes $(B_t)_{t\geq 0}$ with $Cov(B_s, B_t) = \min(s,t)$.

Definition 3
Suppose H a constant with $H \in (0,1]$. Fractional Brownian motion with Hurst index H is a centralized Gaussian processes $(B_t^H)_{t\geq 0}$ with covariance function

$$E\left[B_s^H \cdot B_t^H\right] = Cov\left(B_s^H, B_t^H\right) = \frac{1}{2}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right),\ s,t \geq 0.$$

Definition 3 explain that fractional Brownian Motion has zero expectation, i.e. $E\left[B_t^H\right] = 0$, for all $t \geq 0$. From this, we have $Var\left(B_t^H\right) = t^{2H}$, i.e.

$$\begin{aligned} Var\left(B_t^H\right) &= E\left[\left(B_t^H\right)^2\right] - \left(E\left[B_t^H\right]\right)^2 \\ &= E\left[\left(B_t^H\right)\cdot\left(B_t^H\right)\right] - \left(E\left[B_t^H\right]\right)^2 \\ &= Cov\left(B_t^H, B_t^H\right) - \left(E\left[B_t^H\right]\right)^2 \\ &= \frac{1}{2}\left(t^{2H} + t^{2H} - |t-t|^{2H}\right) - 0 = t^{2H} \end{aligned}$$

Thus, fractional Brownian motion ins a continuous and centered Gaussian process with stationary increments and variance $E\left[\left(B_t^H\right)^2\right] = t^{2H}$. Parameter H allows us to model statistical long range dependence of the log returns. In financial modeling, we assume that $\frac{1}{2} < H < 1$

Proposition 1
If $H = \frac{1}{2}$, then $(B_t^H)_{t\geq 0}$ is Brownian motion.

Proof:
Given $H = \frac{1}{2}$, then

$$\begin{aligned} Cov\left(B_s^{\frac{1}{2}}, B_t^{\frac{1}{2}}\right) &= \frac{1}{2}\left(s^{2\cdot\frac{1}{2}} + t^{2\cdot\frac{1}{2}} - |s-t|^{2\cdot\frac{1}{2}}\right) \\ &= \frac{1}{2}\left(s + t - |s-t|\right) \end{aligned}$$

$$= \begin{cases} \frac{1}{2}(s+t-(s-t)) = \frac{1}{2}\cdot 2t = t & ; s \ge t \\ \frac{1}{2}(s+t-(-(s-t))) = \frac{1}{2}\cdot 2s = s & ; s < t \end{cases}$$
$$= min(s,t).$$

For $t > s$,

$$E\left[\left(B_t^{\frac{1}{2}}\right)\cdot\left(B_s^{\frac{1}{2}}\right)\right] = Cov\left(B_t^{\frac{1}{2}}, B_s^{\frac{1}{2}}\right)$$
$$= \frac{1}{2}\left(t^{2\cdot\frac{1}{2}} + s^{2\cdot\frac{q}{w}} - |t-s|^{2\cdot\frac{1}{2}}\right)$$
$$= \frac{1}{2}(t+s-(t-s)) = s$$

From proposition 1, if $H > \frac{1}{2}$, then $2H > 1$. So $s^{2H} + t^{2H} > |s-t|^{2H}$ or $s^{2H} + t^{2H} - |s-t|^{2H} > 0.$ In another words, Fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ has positive correlation.

The correlation between two *increments* in Fractional Brownian motion show the independent increments. If $H = \frac{1}{2}$, then the correlation between two *increments* is 0. That means, two *increments* are independent.

$$E\left[\left(B_{t+h}^{\frac{1}{2}} - B_t^{\frac{1}{2}}\right)\cdot\left(B_{s+h}^{\frac{1}{2}} - B_s^{\frac{1}{2}}\right)\right]$$
$$= E\left[\left(B_{t+h}^{\frac{1}{2}}\cdot B_{s+h}^{\frac{1}{2}}\right) - B_{t+h}^{\frac{1}{2}}\cdot B_s^{\frac{1}{2}} - B_t^{\frac{1}{2}}\cdot B_{s+h}^{\frac{1}{2}} + \left(B_t^{\frac{1}{2}}\cdot B_s^{\frac{1}{2}}\right)\right]$$
$$= E\left[\left(B_{t+h}^{\frac{1}{2}}\cdot B_{s+h}^{\frac{1}{2}}\right)\right] - E\left[B_{t+h}^{\frac{1}{2}}\cdot B_s^{\frac{1}{2}}\right] - E\left[B_t^{\frac{1}{2}}\cdot B_{s+h}^{\frac{1}{2}}\right] + E\left[\left(B_t^{\frac{1}{2}}\cdot B_s^{\frac{1}{2}}\right)\right]$$
$$= \frac{1}{2}(2(s+h) - 2s - 2(s-h) + 2s) = 0$$

For case $H \ne \frac{1}{2}$, $E\left[\left(B_{t+h}^H - B_t^H\right)\cdot\left(B_{s+h}^H - B_s^H\right)\right] = 0$ not follow. It can show that (Murwaningtyas *et al*., 2016) $E\left[\left(B_{t+h}^H - B_t^H\right)\cdot\left(B_{s+h}^H - B_s^H\right)\right] = \frac{1}{2}h^{2H}\left[(n-1)^{2H} + (n+1)^{2H} - 2n^{2H}\right]$ with $s \le s+h \le t \le t+h$ and $t - s = nh.$

Proposition 2

If $H = 1$, then $B_t^H = tB_1^H$ almost surely for every $t \ge 0.$

Proof:

We can calculate the covariance of the left side dan right side from $B_t^H = tB_1^H$ and show that this covariance is equal.

$$Cov\left(B_t^1, B_t^1\right) = E\left[B_t^1 \cdot B_t^1\right] = \frac{1}{2}\left(t^2 + t^2 - |t-t|^2\right) = t^2$$
$$Cov\left(tB_1^1, tB_1^1\right) = t^2 \cdot Cov\left(B_1^1, B_1^1\right)$$
$$= t^2 \cdot E\left[B_1^1 \cdot B_1^1\right]$$
$$= t^2 \cdot \frac{1}{2}\left(1^2 + 1^2 - |1-1|^2\right) = t^2$$

Another way, we can proof that $B_t^H = tB_1^H$ almost surely for every $t \ge 0.$ by show that $E\left[\left(B_t^H - tB_1^H\right)^2\right] = 0$. That means $B_t^H = tB_1^H$ convergen (almost surely).

Almost surely means:

$$\begin{aligned}
&\left\| B_t^1 - t \cdot B_1^1 \right\|_{L^2(\Re)}^2 = 0 \\
&\quad \Leftrightarrow \int \left(B_t^1 - t \cdot B_1^1 \right)^2 d\mathrm{P} = 0 \\
&\quad \Leftrightarrow E\left[\left(B_t^1 - t \cdot B_1^1 \right)^2 \right] = 0 \\
&\quad \Leftrightarrow B_t^1 = t \cdot B_1^1 \quad \mathrm{P\text{-}almost\text{-}surely.}
\end{aligned}$$

$$\begin{aligned}
E\left[\left(B_t^1 - tB_1^1\right)^2\right] &= E\left[\left(B_t^1 - tB_1^1\right)\cdot\left(B_t^1 - tB_1^1\right)\right] \\
&= E\left[\left(B_t^1\right)^2 - 2tB_1^1B_t^1 + \left(tB_1^1\right)^2\right] \\
&= E\left[\left(B_t^1\right)^2\right] - 2tE\left[B_1^1B_t^1\right] + t^2E\left[\left(B_1^1\right)^2\right] \\
&= \frac{1}{2}\left(t^2 + t^2 - |t-t|^2\right) - 2tE\left[B_1^1B_t^1\right] + t^2\cdot\left(\frac{1}{2}\left(1^2 + 1^2 - |1-1|^2\right)\right) \\
&= t^2 - 2tE\left[B_1^1B_t^1\right] + t^2 \\
&= 2t^2 - 2t\cdot Cov\left(B_1^1, B_t^1\right) \\
&= 2t^2 - 2t\cdot\left(\frac{1}{2}\left(1^2 + t^2 - |1-t|^2\right)\right) \\
&= 2t^2 - 2t\cdot\left(\frac{1}{2}\left(1^2 + t^2 - (1-t)^2\right)\right) \\
&= 2t^2 - 2t\left(\frac{1}{2}\cdot 2t\right) = 0
\end{aligned}$$

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Theorem 1:
If $\left(B_t^H\right)_{t\geq 0}$ is Fractional Brownian motion with Husrt parameter $H \in (0,1]$, then

a. for every $a > 0$, $\left(a^{-H}B_{at}^H\right)_{t\geq 0} \overset{d}{=} \left(B_t^H\right)_{t\geq 0}$ (self-similarity)

b. for every $h > 0$, $\left(B_{t+h}^H - B_h^H\right)_{t\geq 0} \overset{d}{=} \left(B_t^H\right)_{t\geq 0}$ (stationary increments)

Proof:

a. $\left(a^{-H}B_{at}^H\right)_{t\geq 0}$ is centralized Gaussian processes with covariance

$$\begin{aligned}
&Cov\left(a^{-H}B_{at}^H, a^{-H}B_{as}^H\right) \\
&\quad = a^{-2H}\cdot Cov\left(B_{at}^H, B_{as}^H\right) \\
&\quad = a^{-2H}\cdot\left(\frac{1}{2}\left(a^{2H}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right)\right)\right) \\
&\quad = \frac{1}{2}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right)
\end{aligned}$$

b. $\left(B_{t+h}^H - B_h^H\right)_{t\geq 0} \overset{d}{=} \left(B_t^H\right)_{t\geq 0}$ is centralized Gaussian processes with covariance.

$$
\begin{aligned}
&Cov\left(B_{t+h}^H - B_h^H, B_{s+h}^H - B_h^H\right) \\
&= E\left[\left(B_{t+h}^H - B_h^H\right)\cdot\left(B_{s+h}^H - B_h^H\right)\right] \\
&= E\left[\left(B_{t+h}^H \cdot B_{s+h}^H\right) - B_{t+h}^H \cdot B_h^H - B_h^H \cdot B_{s+h}^H + B_h^H \cdot B_h^H\right] \\
&= E\left[B_{t+h}^H \cdot B_{s+h}^H\right] - E\left[B_{t+h}^H \cdot B_h^H\right] - \\
&\quad E\left[B_h^H \cdot B_{s+h}^H\right] + E\left[B_h^H \cdot B_h^H\right] \\
&= \frac{1}{2}\left[(t+h)^{2H} + (s+h)^{2H} - \left|(t+h)-(s+h)\right|^{2H}\right] - \frac{1}{2}\left[(t+h)^{2H} + h^{2H} - \left|(t+h)-h\right|^{2H}\right] - \\
&\quad \frac{1}{2}\left[h^{2H} + (s+h)^{2H} - \left|h-(s+h)\right|^{2H}\right] + \frac{1}{2}\left[h^{2H} + h^{2H} - \left|h-h\right|^{2H}\right] \\
&= \frac{1}{2}\left\{\begin{matrix}\left((t+h)^{2H} + (s+h)^{2H} - |t-s|^{2H}\right) - \left((t+h)^{2H} + h^{2H} - |t|^{2H}\right) - \\ \left(h^{2H} + (s+h)^{2H} - |s|^{2H}\right) + 2h^{2H}\end{matrix}\right\} \\
&= \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right]
\end{aligned}
$$

Theorem 2

For $H \in \left(0, \frac{1}{2}\right)$, Fractional Brownian motions have short-memory and long memory for $H \in \left(\frac{1}{2}, 1\right)$

Proof:

Given Fractional Gaussian noise $X_k = B_k^H - B_{k-1}^H$ and $X_{k+n} = B_{k+n}^H - B_{k+n-1}^H$. The autocovariance function for these noise are

$$
\begin{aligned}
\rho(n) &= Cov\left(X_k, X_{k+n}\right) = E\left[X_k \cdot X_{k+n}\right] \\
&= E\left[\left(B_k^H - B_{k+1}^H\right)\cdot\left(B_{k+n}^H - B_{k+n-1}^H\right)\right] \\
&= E\left[B_1^H\left(B_{n+1}^H - B_n^H\right)\right] = \\
&= Cov\left(B_1^H, B_{n+1}^H\right) - Cov\left(B_1^H, B_n^H\right) \\
&= \frac{1}{2}\left((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right)
\end{aligned}
$$

Because $\lim_{n\to\infty}\frac{\rho(n)}{H(2H-1)n^{2H-2}} = 1$, then $\rho(n) \approx H(2H-1)n^{2H-2}$. If $H < \frac{1}{2}$, then $\rho(n) < 0$ and $\sum_n \rho(n) < \infty$ (short memory). If $H > \frac{1}{2}$, then $\rho(n) > 0$ and $\sum_n \rho(n) = \infty$ (long memory). The long memory means X_k and X_{k+n} decrease very slow for long time.

6. Conclusions

We show that the Brownian motion is a special case from Fractional Brownian motion with Hurst index = ½. If Hurst index > ½. we can modelling stock prices use Fractional Brownian motion for long memory. Fractional Brownian motions have two important properties: self-similarity and stationary increments.

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