An Inventory Model with Linear Demand Rate, Finite Rate of Production with Shortages and Complete Backlogging

C. K. Sahoo and S. K. Sahoo
Institute of Mathematics and Application
Bhubaneswar-751007, Orissa, India

Abstract

This paper deals in developing inventory model with linear demand rate allowing shortages in the inventory. These shortages are considered to be completely backlogged. We have assumed that the production rate is finite and proportional to the demand rate. The analytical solution of the model has been done to obtain the optimal solution of the problem. Suitable numerical example has been discussed to understand the problem. Further we have made sensitivity analysis of the optimal solution with respect to the changes in the values of the system parameters. This model is suitable in case of steady increase or decrease in the demand in the market for some products.

Keywords
Inventory; Economic order quantity; demand; Shortage.

1. Introduction

In traditional inventory model the demand rate of the item was assumed to be constant which is not generally to in real life situation. The first modification for varying demand was suggested by have Silver and Meal [11] after which a lot of work been done by many researchers such as Silver and Meal [12] developed an approximate solution procedure, known as the Silver-Meal heuristic for general case of a deterministic, time-dependent demand pattern, for the first time, the classical no-shortage inventory policy for the case of linear, time-dependent demand, significant contributions in this direction has been due to researchers namely Ritchi[10], Shortages and backlogging has also been consider along with varying demand in course of time by many researchers such as Deb and Chaudhuri [4] was the first to incorporate shortages into the inventory lot sizing with a linearly problem linearly increasing time-varying demand.

EOQ models for deteriorating items with trended demand have also considered by several researchers like Bhari – Kashani [2], Goswami and Chodhuri [6] Chung and Ting [3], Hariga [7], Giri and Chakrabarty [5], Jalan and Chodhuri [8] and Lin et al. [9].A group of researchers have also devoted their attention to inventory replenishment problems with exponentially time-varying demand patterns. Some of the contributions in this direction have come from Agrawal and Bhari – Kashani [1], etc.

In the present paper, we assume that time- dependence of demand follows a linear demand. Also the production rate is assumed to be finite and proportional to the demand rate. Shortages are allowed and are completely backlogged. An analytical solution not the mode is discussed and illustrated with the help of numerical examples. Sensitivity of the optimal solution with respect to changes in different parameter values is also examined.

Assumption

1. The demand rate at any time ‘t’ is : $R(t) = \alpha t \qquad \alpha > 0$
2. The production rate is $K(t) = \lambda R(t)$ where $(\lambda < 1)$ a constant is also.Therefore $K(t) > R(t)$.
3. The on-hand inventory does not deteriorate with time.
4. Lead time is zero.
5. Shortages are allowed and are completely backlogged.

Notations

c_1 Carrying cost per unit per unit time.

c_2 Shortage cost per unit per unit time.

c_3 Setup cost per production run

c_1, c_2, c_3 are all assumed to be known and fixed during production cycle.
'C' the total average cost for a production cycle.

2. Mathematical Model

The inventory level at different instants of time is initially (i.e. at time $t = 0$, the stock level is zero. The shortage starts at $t = 0$ and accumulates up to the level P at $t = t_1$. The production starts at $t = t_1$ and the backlog is cleared at $t = t_2$. The stock level attains a level S at $t = t_4$, when production is stopped. The inventory level gradually decreases due to demand and becomes zero at $t = t_4$. The cycle then repeats itself after time t_4. Our problem is to determine the optimum values of 'S', 'P' and 'C'. Now if $Q(t)$ be the instantaneous inventory level at any time t $(0 \le t \le t_4)$, the differential equations describing the instantaneous states of $Q(t)$ in the interval $(0, t_4)$ are

$$\frac{dQ}{dt} = -R(t) \qquad 0 \le t \le t_1 \qquad (1)$$

$$\frac{dQ}{dt} = K(t) - R(t) \qquad t_1 \le t \le t_2 \qquad (2)$$

$$\frac{dQ}{dt} = K(t) - R(t) \qquad t_2 \le t \le t_3 \qquad (3)$$

$$\frac{dQ}{dt} = K(t) - R(t) \qquad t_3 \le t \le t_4 \qquad (4)$$

with the boundary conditions

$$Q(0) = 0,\ Q(t_1) = -P,\ Q(t_2) = 0,\ Q(t_3) = S,\ Q(t_4) = 0. \qquad (5)$$

Now substituting $R(t) = \alpha t$ and $K(t) = \lambda R(t)$ in the equations (1) – (4) are solving them using the boundary conditions (5), we get the solutions as follows:

$$Q(t) = -\left(\frac{\alpha}{2}\right)t^2 \qquad 0 \le t \le t_1 \qquad (6)$$

$$Q(t) = -\frac{\alpha(\lambda - 1)}{2}(t^2 - t_2^2) \qquad t_1 \le t \le t_2 \qquad (7)$$

$$Q(t) = \frac{\alpha(\lambda - 1)}{2}(t^2 - t_2^2) \qquad t_2 \le t \le t_3 \qquad (8)$$

$$Q(t) = \frac{\alpha}{2}(t_4{}^2 - t^2) \qquad t_3 \le t \le t_4 \qquad (9)$$

Using the condition in $Q(t) = -P$ in (6), we get

$$P = \left(\frac{\alpha}{2}\right)t_1{}^2 \qquad (10)$$

Similarly using the condition $Q(t_1) = -P$ in (7), we get

$$P = -\frac{\alpha(\lambda - 1)}{2}(t_1{}^2 - t_2^2) \qquad (11)$$

Equating these two values of P, we get

$$t_1 = t_2\left(1 - \frac{1}{\lambda}\right)^{\frac{1}{2}} \qquad (12)$$

Again, using the condition in $Q(t_3) = S$ (8) and (9) we get respectively

$$S=\frac{\alpha(\lambda-1)}{2}(t_3{}^2-t_2^2) \tag{13}$$

and

$$S=\frac{\alpha}{2}(t_4{}^2-t_3^2) \tag{14}$$

Equating these two values of 'S' we get

$$t_3=\left[\frac{t_4{}^2+t_2{}^2(\lambda-1)}{\lambda}\right]^{\frac{1}{2}} \tag{15}$$

Now we found the different the costs involved in the system.
The total shortage cost in the system is

$$SC=c_2\left[\int_0^{t_1}-Q(t)dt+\int_{t_1}^{t_2}-Q(t)dt\right]$$

$$=c_2\left[\frac{\alpha}{6}\times t_1^3-\frac{\alpha(\lambda-1)}{2}\left\{\frac{t_2^3-t_1^3}{3}-t_2^2(t_2-t_1)\right\}\right] \tag{16}$$

The total inventory holding cost in the system is

$$HC=c_1\left[\int_{t_2}^{t_3}Q(t)dt+\int_{t_3}^{t_4}Q(t)dt\right]$$

$$=c_1\left[\frac{\alpha(\lambda-1)}{2}\left\{\frac{t_3^3-t_2^3}{3}-t_2{}^2(t_3-t_2)\right\}+\frac{\alpha}{2}\left\{t_4{}^2(t_4-t_3)-\frac{t_4^3-t_3^3}{3}\right\}\right] \tag{17}$$

Therefore the average cost of the system is

$$C=\frac{1}{t_4}\left(SC+HC+c_3\right)$$

$$=\frac{1}{t_4}\left[\; c_2\left[\;\frac{\alpha}{6}\times t_1^3-\frac{\alpha(\lambda-1)}{2}\left\{\frac{t_2^3-t_1^3}{3}-t_2^2(t_2-t_1)\right\}\;\right]+\right.$$

$$\left. c_1\left[\;\frac{\alpha(\lambda-1)}{2}\left\{\frac{t_3^3-t_2^3}{3}-t_2{}^2(t_3-t_2)\right\}+\frac{\alpha}{2}\left\{t_4{}^2(t_4-t_3)-\frac{t_4^3-t_3^3}{3}\right\}\;\right]+c_3\;\right] \tag{18}$$

Substituting the values of t_1 from (12) and t_3 from (15), C becomes a function of the variables t_1 and t_4. Therefore, C will be minimum if

$$\frac{\partial C}{\partial t_2}=0 \quad , \quad \frac{\partial C}{\partial t_4}=0\text{, Provided}\quad \frac{\partial^2 C}{\partial t_2{}^2}\times\frac{\partial^2 C}{\partial t_4{}^2}-\left(\frac{\partial^2 C}{\partial t_2\partial t_4}\right)^2>0 \tag{19}$$

From (18) and (19), we get the equations

$$\frac{1}{t_4}\left[\quad c_2\left[\frac{\alpha}{2}\left(\frac{\lambda-1}{\lambda}\right)^{\frac{3}{2}}t_2{}^2-\alpha\left(\frac{\lambda-1}{2}\right)\times\left\{t_2^2\left[1-\left(\frac{\lambda-1}{\lambda}\right)^{\frac{3}{2}}\right]-3t_2^2\left[1-\left(\frac{\lambda-1}{\lambda}\right)^{\frac{1}{2}}\right]\right\}\right]\right.$$

$$+c_1\Big[\frac{\alpha(\lambda-1)}{2}\times\left\{\frac{t_4^2+t_2^2(\lambda-1)}{\lambda}\right\}^{\frac{1}{2}}\times t_2\left(\frac{\lambda-1}{\lambda}\right)+2t_2^{\,2}-2t_2\left\{\frac{t_4^2+t_2^2(\lambda-1)}{\lambda}\right\}^{\frac{1}{2}}-t_2^3\times\left\{\left\{\frac{t_4^2+t_2^2(\lambda-1)}{\lambda}\right\}^{-\frac{1}{2}}\right\}$$

$$\times\left(\frac{\lambda-1}{\lambda}\right)\Big]+\frac{\alpha}{2}\Big[-t_4^2\times\left\{\frac{t_4^2+t_2^2(\lambda-1)}{\lambda}\right\}^{-\frac{1}{2}}t_2\left(\frac{\lambda-1}{\lambda}\right)+\left\{\frac{t_4^2+t_2^2(\lambda-1)}{\lambda}\right\}^{\frac{1}{2}}t_2\left(\frac{\lambda-1}{\lambda}\right)\Big]\Big]\Big]=0 \quad (20)$$

and $$\frac{1}{t_4}\Big[c_1\times\frac{\alpha(\lambda-1)}{2}\Big[\Big[\left\{\frac{t_4^2+t_2^2(\lambda-1)}{\lambda}\right\}^{\frac{1}{2}}\times\frac{1}{\lambda}\times t_4-t_2^2\left\{\frac{t_4^2+t_2^2(\lambda-1)}{\lambda}\right\}^{-\frac{1}{2}}\times\frac{1}{\lambda}\times t_4\Big]$$

$$\frac{\alpha}{2}\Big[3t_4^2-t_4^2\left\{\frac{t_4^2+t_2^2(\lambda-1)}{\lambda}\right\}^{-\frac{1}{2}}\times\frac{1}{\lambda}\times t_4-2t_4\left\{\frac{t_4^2+t_2^2(\lambda-1)}{\lambda}\right\}^{\frac{1}{2}}-t_4^2+\left\{\frac{t_4^2+t_2^2(\lambda-1)}{\lambda}\right\}^{\frac{1}{2}}\times\frac{1}{\lambda}\times t_4\Big]\Big]\Big]$$

$$-\frac{1}{t_4^2}\Big[c_2\Big[\frac{\alpha}{6}\times\left(\frac{\lambda-1}{\lambda}\right)^{\frac{3}{2}}t_2^3-\frac{\alpha(\lambda-1)}{2}\Big[\frac{t_2^3}{3}-\frac{t_2^3}{3}\left(\frac{\lambda-1}{\lambda}\right)^{\frac{3}{2}}-t_2^2\left\{t_2-t_2\left(\frac{\lambda-1}{\lambda}\right)^{\frac{1}{2}}\right\}\Big]\Big]\Big]$$

$$+c_1\Big[\frac{\alpha(\lambda-1)}{2}\Big[\frac{1}{3}\left\{\frac{t_4^2+t_2^2(\lambda-1)}{\lambda}\right\}^{\frac{3}{2}}-\frac{t_2^3}{3}-t_2^2\Big[\left\{\frac{t_4^2+t_2^2(\lambda-1)}{\lambda}\right\}^{\frac{1}{2}}-t_2\Big]\Big]$$

$$+\frac{\alpha}{2}\Big[t_4^2\Big[t_4-\left\{\frac{t_4^2+t_2^2(\lambda-1)}{\lambda}\right\}^{\frac{1}{2}}\Big]-\frac{t_4^3}{3}-\frac{1}{3}\times\left\{\frac{t_4^2+t_2^2(\lambda-1)}{\lambda}\right\}^{\frac{3}{2}}\Big]+c_3\Big]=0 \quad (21)$$

3. Numerical Examples

Let λ = 2.5, c_1 = 20, c_2 = 30, c_3 = 40 in appropriate units. From (21) and (22), we obtain the optimum values of t_i (i=1, 2, 3, 4). Taking one parameters used in model are analyzed in the following table. suppose α =200, the optimum values of t_i (i=1, 2, 3, 4) are $t_1 = 0.30924$, t_2 = 0.399245, t_3= 0.535498, t_4= 0.691232. Substitute these optimum values of t_1, t_2, t_3 and t_4 in equation (18) using Mathematica 5.1, we get the optimum average cost C* = 918.032. The optimum values of P and S obtained from (10) and (14) respectively S* = 19.1044 and P*= 9.5638.

Table

Changing Parameter	%change system	change in t_1	change t_2	change in t_3	change in t_4	change in C
	+10	0.294644	0.294644	0.51987	0.667497	918.385
	+20	0.281981	0.281981	0.496818	0.64752	918.641
	+30	0.270889	0.270889	0.4890993	0.628437	919.018
	+40	0.261082	0.261082	0.466951	0.612126	919.442
α	+50	0.252343	0.252343	0.45439	0.597492	19.911
	-10	0.326325	0.326325	0.559561	0.718715	917.792
	-20	0.346579	0.346579	0.587979	0.751002	917.579
	-30	0.37107	0.37107	0.622168	0.78962	917.367
	-40	0.4014	0.4014	0.664279	0.836877	917.098
	-50	0.440172	0 .440172	0.717793	0.896487	916.662
	+10	0.286011	0.358534	0.481547	0.642444	902.43
	+20	0.264727	0.324223	0.435858	0.599727	889.007
	+30	0.245409	0.294945	0.396759	0.562026	877.537
	+40	0.227959	0.269725	0.363017	0.52854	867.774

λ	+50	0.212232	0.247834	0.33369	0.498647	859.472
	-10	0.334232	0.44841	0.600108	0.747625	935.881
	-20	0.360301	0.509543	0.679135	0.814135	955.786
	-30	0.385951	0.58955	0.779412	0.895778	977.006
	-40	0.407776	0.706289	0.917045	1.006	997.967
	-50	0.417269	0.933041	0.115536	1.20455	997.967
	+10	0.30181	0.389635	0.515932	0.66162	9910809
	+20	0.295461	0.381438	0.499274	0.636353	1066.64
	+30	0.289964	0.374342	0.4889	0.614487	1142.27
	+40	0.285146	0.368122	0.472314	0.59534	1218.52
c_1	+50	0.28088	0.362614	0.461209	0.578404	1295.27
	-10	0.31814	0.410717	0.558891	0.726544	845.69
	-20	0.328997	0.424733	0.58748	0.769567	775.379
	-30	0.342661	0.4424373	0.623417	0.823456	708.089
	-40	0.360561	0.465482	0.670316	0.893476	645.592
	-50	0.38537	0.49751	0.734798	0.989214	591.417
	+10	0.317116	0.40935	0.556226	0.722535	937.187
	+20	0.324832	0.419356	0.576575	0.75399	957.755
	+30	0.332431	0.429166	0.596609	0.783314	997.773
	+40	0.339938	0.438858	0.616381	0.812969	1003.28
c_2	+50	0.347374	0.448458	0.635934	0.842234	1028.33
	-10	0.301207	0.388857	0.514315	0.659155	900.255
	-20	0.29293	0.378171	0.492578	0.626149	883.827
	-30	0.284361	0.367108	0.470162	0.592013	868.722
	-40	0.275419	0.355564	0.446904	0.556479	854.922
	-50	0.265992	0.343394	0.422575	0.519177	842.414
	+10	0.324654	0.719127	0.557211	0.716037	1002.64
	+20	0.339418	0.438187	0.577947	0.739624	1093.49
	+30	0.353603	0.4565	0.597802	0.762122	1184.33
	+40	0.36726	0.47413	0.616859	0.783639	1275.16
c_3	+50	0.380429	0.491132	0.635186	0.804266	1365.97
	-10	0.293148	0.378453	0.512695	0.665056	821.001
	-20	0.276258	0.356648	0.488661	0.63732	730.245
	-30	0.261055	0.333702	0.463223	0.607786	639.585
	-40	0.239704	0.309457	0.436163	0.576151	548.994
	-50	0.229889	0.296186	0.421938	0.559425	463.762

4. Sensitivity Analysis

To study the effects of changes in the system parameters, c_1, c_2 , c_3 , α , and λ on the optimal cost derived by the proposed method, sensitivity analysis is performed by changing (increasing or decreasing) the parameters by -50 % and 50 % and taking one parameter at a time, keeping the remaining parameters at their original values.

On the basis of the results of table, the following observation can be made.

(i) Decrease in the value of either of the parameters α , c_1 then t_1^* , t_2^*, t_3^*, t_4^* is increased and C^* is decreased.

(ii) Decrease in the values of either of the parameters c_2, c_3 then t_1^* , t_2^*, t_3^*, t_4^* and C^* is decreased.

(iii) Decrease in the value of the parameter λ then t_1^* , t_2^* , t_3^* , t_4^* and C^* is increased.

5. Conclusion

In this paper we assumed that time dependent linear demand and shortages are completely backlogged. Here the production rate assumed to be finite proportional to the demand and shortage with completely backlogged. In real market situations, demand is unlikely to vary with a rate which is so high as exponential. Time-dependence of demand is usually nonlinear in nature. The advantage of the linear functional form of the demand take care of steady increasing or steady decreasing and constant demand for different ranges of values of its parameter.

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