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A Genetic Algorithm for Fuzzy All-Pairs Shortest Path in a Network

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Abstract

The shortest path problem is an important classical network optimization problem arising from many applications. In many cases, however, some aspects of a network-theoretic problem may be uncertain. For example, the vehicle travel time or vehicle capacity on a road network may not be known exactly. In such cases, it is natural to deal with the uncertainty using fuzzy set theory. Here, we consider solving the All-Pairs Shortest Paths Problem (APSPP) in a network with fuzzy arc lengths. Making use of a suitable ranking method, we propose a new approach to solve the fuzzy APSPP using a genetic algorithm. Illustrative examples are worked out to demonstrate our proposed method.

Keywords
Network, All-pairs shortest path problem, Genetic algorithm, Fuzzy sets

1. Introduction

The shortest path (SP) problem has received much attention in the literature. It is important to many applications such as communication, transportation, scheduling and routing. In a network, the arc length may represent time or cost. Therefore, in real applications, it can be considered to be a fuzzy set. Fuzzy set theory, proposed by Zadeh [27], is frequently utilized to deal with uncertainties in a problem. We consider a directed network consisting of a finite set of vertices and a finite set of directed arcs. It is assumed that there is only one directed arc between any two vertices. We specify a source vertex and a destination vertex. Each arc length is represented by a fuzzy number, and the length of a path is defined to be the fuzzy sum of arc lengths along the path. We are concerned with finding a path from the source vertex to any other vertex while optimizing the length of the path.

Blue et al. [2] give taxonomy of graph fuzziness that distinguishes five basic types combining fuzzy or crisp vertex sets with fuzzy or crisp edge sets and fuzzy weights and fuzzy connectivity.

Numerous papers have been published for solving fuzzy graph problems [11, 13, 16, 19, 25]. The fuzzy shortest path problem was first analyzed by Dubois and Prade [8]. They utilized the algorithms [10, 12] proposed by Floyd and Ford, to treat the fuzzy shortest path problem. Klein [15] proposed a dynamic programming recursion-based fuzzy algorithm. Lin and Chen [17] found the fuzzy shortest path length in a network by means of a fuzzy linear programming approach. Another algorithm for this problem was presented by Okada and Gen [23, 24], using a generalization of Djikstra's algorithm. In this algorithm, the weights of the arcs were considered to be interval numbers and defined by a partial order of interval numbers. Okada and Soper [22] proposed a fuzzy algorithm, which was based on multiple labeling methods to offer non-dominated paths to the decision maker. Blue et al. [2] presented an algorithm for finding a cut value to limit the number of analyzed paths, and then applied a modified version of the *k*-shortest path (crisp) algorithm proposed by Eppstein [9]. Okada [21] introduced the concept of the degree of possibility of an arc being on the shortest path. Among the most recent work is the paper by Nayeem and Pal [20] that proposed an algorithm based on the acceptance index introduced by Sengupta and Pal [26] giving a single fuzzy shortest path or a guideline for choosing the best fuzzy shortest path according to the decision-maker's viewpoint. Chuang and Kung [5] proposed a fuzzy shortest path length procedure that could find a fuzzy shortest path length among all possible paths in a network. It is based on the idea that a crisp number is minimal if and only if any other number is larger than or equal to it. Hernendes et al. [11] proposed an iterative algorithm assuming a generic ranking index for comparing the fuzzy numbers involved in the problem. This algorithm is based on the Ford–Moore–Bellman algorithm for classical graphs. Mahdavi et al. [18] proposed a dynamic programming approach to solve the fuzzy shortest chain problem using a suitable ranking method.

This situation is a motivation for us to propose an alternative approach to deal with fuzzy shortest path problem instead. Since the fuzzy min operator based on the extension principle leads to non-dominated solutions, we propose a new approach to solve the fuzzy APSPP using a suitable fuzzy ranking method and Genetic Algorithm (GA).

The remainder of the paper is organized as follows. In Section 2, some elementary concepts and related operations of fuzzy set theory are provided. In Section 3, a mathematical formulation of the fuzzy all-pairs shortest path problem is given. There, we also discuss how to compare fuzzy distances. In Section 4, we present a genetic algorithm to find the fuzzy all-pairs shortest path length. In Section 5, an example is worked out to demonstrate the effectiveness of the algorithm. Finally, conclusions are given in Section 6.

2. Definition of directed graph

A directed graph is a network $G = (N, A)$ composed of a finite set of nodes N and a set of directed arcs A; an arc is denoted by an ordered pair (i, j), where $i \neq j$. To represent a directed graph, we make use of an adjacency matrix.

Definition: A matrix $A = [a_{ij}]_{n\times n}$ is called the adjacency matrix in a directed network $G = (N, A)$, if

$$a_{ij} = \begin{cases} 1 & if \quad (i, j) \in A \\ 0 & if \quad (i, j) \notin A . \end{cases} \tag{1}$$

We denote the *i th* row of A by $a(i)$ and the set of ending nodes of outgoing arcs from node *i* by

$$a^1(i) = \{ j \mid (i, j) \in A \ , \ a_{ij} = 1 \}.$$

3. The Formulation of the Problem

Consider a directed graph $G = (N, A)$. Let l_{ij} be the length (or weight) associated with arc (i, j). A path from a node $i \in N$ to a node $j \in N$ is a sequence of nodes $i, i_1, i_2, \ldots , i_n, j$ such that $(i, i_1), (i_1, i_2), \ldots ,$ (i_n, j) are arcs in A. The length of a path is the sum of the lengths of the arcs on the path. The objective in the all-pairs shortest path problem is to find a shortest path between all pairs of nodes.

In the fuzzy all-pairs shortest path problem (FAPSPP), we need to find minimal values amongst fuzzy lengths. Thus, we should have a method of ranking the fuzzy numbers for comparison purposes. An ordering relation $\preceq$ of fuzzy trapezoidal numbers can be defined as follows [16]:

$$\tilde{A} \preceq \tilde{B} \iff (a_1 \le b_1) \wedge (a_2 \le b_2) \wedge (a_3 \le b_3) \wedge (a_4 \le b_4) \cdot \tag{2}$$

However, this relation is not a complete ordering, as fuzzy numbers $\tilde{A}$ and $\tilde{B}$ satisfying

$$(\exists i, j \in \{1,2,3,4\}) : (a_i < b_i) \wedge (a_i > b_i) \tag{3}$$

are not comparable by $\preceq$.

This is a difficulty with comparison of fuzzy numbers. For this reason, the ranking or ordering methods of fuzzy quantities have been proposed by many authors. Unfortunately, none of these methods is universally accepted.

Here, we make use of a new ranking method for fuzzy numbers proposed by Mahdavi et al. [18].

Define the fuzzy min operation similar to the fuzzy addition as follows:

$$Min(\tilde{a}, \tilde{b}) = (\min(a_1, b_1), \min(a_2, b_2), \min(a_3, b_3), \min(a_4, b_4)). \tag{4}$$

It is evident that, for non-comparable fuzzy numbers $\tilde{a}$ and $\tilde{b}$, this fuzzy min operation results in a fuzzy number different from both $\tilde{a}$ and $\tilde{b}$. To avert this drawback, Mahdavi et al. [18] proposed using a new method based on distance between fuzzy numbers, given in [25].

Definition 3-1. The $D_{p,q}$ -distance, indexed by parameters $p, 1 < p < \infty$, and q , $0 < q < 1$, between two fuzzy numbers $\tilde{a}$ and $\tilde{b}$ is a nonnegative function given as follows:

$$D_{p,q}(\tilde{a},\tilde{b})=\begin{cases}\left[(1-q)\int_0^1\left|a_\alpha^- - b_\alpha^-\right|^p d\alpha + q\int_0^1\left|a_\alpha^+ - b_\alpha^+\right|^p d\alpha\right]^{\frac{1}{p}}, & p<\infty,\\ (1-q)\sup_{0<\alpha\le 1}\left(\left|a_\alpha^- - b_\alpha^-\right|\right)+q\inf_{0<\alpha\le 1}\left(\left|a_\alpha^+ - b_\alpha^+\right|\right), & p=\infty.\end{cases} \tag{5}$$

The analytical properties of $D_{p,q}$ depend on the first parameter *p*, while the second parameter *q* of $D_{p,q}$ characterizes the subjective weight attributed to the end points of the support of the fuzzy numbers. If there is no reason for distinguishing any side of the fuzzy numbers, then $D_{2,\frac{1}{2}}$ is recommended.

For triangular fuzzy numbers $\tilde{a}=(a_1,a_2,a_3)$ and $\tilde{b}=(b_1,b_2,b_3)$, the above distance with $p=2$ and $q=\frac{1}{2}$ is calculated to be:

$$D_{2,\frac{1}{2}}\left(\tilde{a},\tilde{b}\right)=\sqrt{\frac{1}{6}\left[\sum_{i=1}^{3}(b_i-a_i)^2+(b_2-a_2)^2+\sum_{i\in\{1,2\}}(b_i-a_i)(b_{i+1}-a_{i+1})\right]} \tag{6}$$

and if $\tilde{a}=(a_1,a_2,a_3,a_4)$ and $\tilde{b}=(b_1,b_2,b_3,b_4)$ are trapezoidal fuzzy numbers, then the distance is calculated as:

$$D_{2,\frac{1}{2}}\left(\tilde{a},\tilde{b}\right)=\sqrt{\frac{1}{6}\left[\sum_{i=1}^{4}(b_i-a_i)^2+\sum_{i\in\{1,3\}}(b_i-a_i)(b_{i+1}-a_{i+1})\right]}. \tag{7}$$

Now, we make use of the distance function (7) to acquire the distances of $\tilde{a}$ and $\tilde{b}$ to $MV=Min(\tilde{a},\tilde{b})$, as obtained by (4). We get $D_{2,\frac{1}{2}}(\tilde{a},MV)=0.1667$ and $D_{2,\frac{1}{2}}(\tilde{b},MV)=1.33$. Thus, $\tilde{a}$, having a lower distance from MV, is decided to be smaller than $\tilde{b}$.

4. The Genetic Algorithm

Genetic Algorithm (GA) is based on an analogy to the phenomenon of natural selection in biology. First, a chromosome structure is defined to represent the solutions of the problem. A GA can be implemented in a variety of ways. The excellent books by Goldberg [6] and Davis [7] describe many possible variants of GAs.

4.1. GA for solving shortest path problem with fuzzy arc lengths

Here, we describe the encoding scheme used by the GAs. Genetic operators specific to this encoding scheme are also defined. These include the initialization, crossover and mutation operators.

4.1.1. GA Encoding Schemes

How to encode a path in a graph is critical for developing a genetic algorithm to this problem. this is not as easy as traveling salesmen problem to find out a natural representation. Special difficulties arise from (a) a path contains a variable number of nodes and the maximal number is *n* – 1 for an *n* node graph, and (b) a random sequence of edges usually does not correspond to a path. To overcome such difficulties, we adopted an indirect approach: encode some guiding information for constructing a path, but a path itself, in a chromosome. The path is generated by the sequential node appending procedure beginning with the specified node 1 and terminating at the specified node *n*. A vector *p* is used to keep the trace of the nodes in the path.

Algorithm1 (generating initial population)

1. Find the vicinity matrix of directed network *G* = (*N, A*), determine *pop-size* and set $k\leftarrow 1$.
2. Put $i=1$, $l=1$ and $p(l)=1$. .
3. Select a member of $a^1(i)$ and call it j. Let $l=l+1$ and $p(l)=j$. .
4. If $j\neq n$ then let $i\leftarrow j$ and go to (3).
5. Find produced path by the labels in *P* and let $k\leftarrow k+1$.
6. If $k\le pop-size$ then go to 2 else stop.

4.1.2. Crossover operator
Crossover combines information from two parents such that the two children have a “resemblance” to each parent. Standard crossovers such as one-point, two-point, and uniform are used in GA models. Two paths, called parents, are randomly selected from the population. Then, we select one or two common members (genes) and replace different sections of the codes for parents. Hence, two new children (chromosomes) are generated. It is obvious that the generated paths are feasible.

4.1.3. Mutation operator
The traditional mutation operator mutates the gene's value randomly according to a small probability of mutation; thus, it is merely a random walk and does not guarantee a positive direction toward the optimal solution. The proposed heuristic mutation remedies this deficiency. In this scheme, the number of paths or selected chromosomes, q, for the mutation operator is determined using a mutation operator’s rate or probability(p_m); the number of paths q is computed to be the product of population size and p_m, and q different random numbers are generated between 1 and population size to be used as the number of selected paths. Then, a number, say r, between 1 and path length is selected randomly. The components 1 to r-1 is kept unchanged, and components r to path length are removed, replacing them with new nodes obtained by using Algorithm 1.

4.1.3 Evaluation and selection strategy
Considering population chromosome, a path for each chromosome is determined and the path length is considered as the chromosome value. Then, for comparison of chromosome values, we use (4) and $D_{2,\frac{1}{2}}$ distance function as explained before. The new algorithm for finding shortest path follows here.

Algorithm 2 (finding shortest path)

Step 1. Find the possible paths from source vertex s to destination vertex d from among produced population in each repetition and compute the corresponding path lengths L_i, $i = 1,2,. . . ,m$, for the possible m paths.

Step 2. Find the fuzzy shortest length $\tilde{L}^{\min}$ by the following steps:

Step 2-1. Set $\tilde{L}_{\min} = \tilde{L}_1$.

Step 2-2. For $i = 2,...,m$ do

begin

Calculate

$\tilde{M}V = Min(\tilde{L}_{\min}, \tilde{L}_i)$ by Eq. (4).

Find the distance $D_{p,q}$ of

$\tilde{M}V$ from $\tilde{L}_{\min}$ and $\tilde{L}_i$ using

Eq. (7):

$D_1 = D_{p,q}(\tilde{M}V, \tilde{L}_{\min})$

$D_2 = D_{p,q}(\tilde{M}V, \tilde{L}_i)$

$\tilde{L}_{\min} = \arg\min(\tilde{D}_1, \tilde{D}_2)$

end

In our experiment, the roulette wheel approach, which was adopted as the selection procedure, is one of the fitness-proportional selections, and the elitist method was combined with this approach in order to preserve the best chromosome for the next generation and overcome the stochastic errors of sampling. With the elitist selection, if the best individual in the current generation is not reproduced into the new generation, one individual is randomly removed from the new population and the best one is added to the new population.

5. Numerical Illustration
An example: Consider a mobile service company which handles 40 geographical centers. A configuration of a telecommunication network is presented in Table 1. Assume that the distance between any two centers is a trapezoidal fuzzy number. The company wants to find a shortest path for an effective message flow amongst the centers.

In the example, number of repetition is 100, number of chromosome is 20, crossover ratio is 0.4 and mutation ratio is 0.1. Figure 1 shows the convergence curve for the example.

Table 1. The arc lengths of the network for the example

Arc	Arc length	Arc	Arc length
(1,2)	(12,13,15,17)	(1,3)	(9,11,13,15)
(1,4)	(8,10,12,13)	(1,5)	(7,8,9,10)
(2,6)	(5,10,15,16)	(2,7)	(6,11,11,13)
(3,8)	(10,11,16,17)	(4,7)	(17,20,22,24)
(4,11)	(6,10,13,14)	(5,8)	(6,9,11,13)
(5,11)	(7,10,13,14)	(5,12)	(10,13,15,17)
(6,9)	(6,8,10,11)	(6,10)	(10,11,14,15)
(7,10)	(9,10,12,13)	(7,11)	(6,7,8,9)
(8,12)	(5,8,9,10)	(8,13)	(3,5,8,10)
(9,16)	(6,7,9,10)	(10,16)	(12,13,16,17)
(10,17)	(15,19,20,21)	(11,14)	(8,9,11,13)
(11,17)	(6,9,11,13)	(12,14)	(13,14,16,18)
(12,15)	(12,14,15,16)	(13,15)	(10,12,14,15)
(13,19)	(17,18,19,20)	(14,21)	(11,12,13,14)
(15,18)	(8,9,11,13)	(15,19)	(5,7,10,12)
(16,20)	(9,12,14,16)	(17,20)	(7,10,11,12)
(17,21)	(6,7,8,10)	(18,21)	(15,17,18,19)
(18,22)	(3,5,7,9)	(18,23)	(5,7,9,11)
(19,22)	(15,16,17,19)	(20,23)	(13,14,16,17)
(21,23)	(12,15,17,18)	(22,23)	(4,5,6,8)
(31,40)	(3,4,5,6)	(25,40)	(4,5,6,7)
(21,39)	(8,9,11,13)	(20,39)	(4,5,6,7)
(36,38)	(12,14,14,15)	(33,37)	(2,2,3,4)
(14,37)	(1,2,3,12)	(31,36)	(4,5,6,7)
(20,35)	(5,6,7,8)	(12,35)	(2,3,4,4)
(19,34)	(3,4,4,4)	(38,40)	(5,6,7,8)
(24,33)	(11,11,12,13)	(39,40)	(4,5,6,9)
(31,32)	(3,4,5,8)	(37,39)	(7,8,9,10)
(26,32)	(5,6,7,8)	(26,31)	(3,4,5,6)
(24,31)	(6,7,8,11)	(35,36)	(3,3,5,7)
(34,36)	(3,4,5,5)	(28,30)	(6,7,8,8)
(14,30)	(2,3,4,5)	(34,38)	(5,6,6,7)
(21,29)	(6,7,8,9)	(26,29)	(3,4,4,6)
(27,28)	(6,8,9,10)	(13,28)	(1,4,5,6)
(32,34)	(4,5,5,9)	(26,27)	(6,6,7,8)
(15,27)	(3,4,9,12)	(32,33)	(7,8,8,9)
(30,40)	(3,4,5,8)	(22,26)	(2,7,8,9)
(24,26)	(1,4,5,6)	(16,25)	(2,3,8,10)
(30,36)	(1,4,5,6)	(17,25)	(4,5,8,9)
(29,31)	(2,4,6,8)	(23,25)	(2,3,8,10)
(23,24)	(1,6,7,10)		

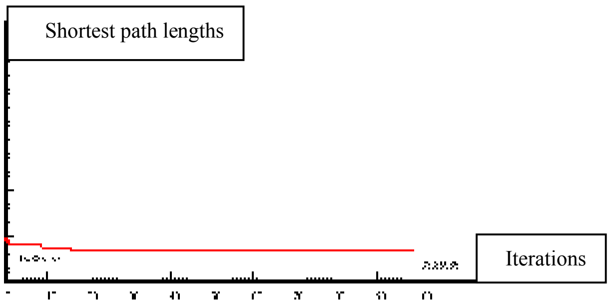

Figure. 1. Convergence curve for the example.

6. Conclusions

We considered the problem of finding the all-pairs fuzzy shortest paths, with the lengths of arcs given by trapezoidal fuzzy numbers. First, we proposed an order relation between fuzzy numbers. Then, we developed a fuzzy ranking method to avoid generation of the set of non-dominated paths (or Pareto optimal paths) because the number of non-dominated paths derived from a large network can be too numerous, and it could be difficult for a decision maker to choose a preferable path. Then, we investigated the possibility of using genetic algorithm to solve fuzzy shortest path problems. The proposed approach illustrated by being tested on a randomly generated problem having 40 nodes and 81 edges.

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