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A Genetic Algorithm for Fuzzy Shortest Path in a Network with mixed fuzzy arc lengths

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Abstract

We are concerned with the design of a model and an algorithm for computing a shortest path in a network having various types of fuzzy arc lengths. First, we develop a new technique for the addition of various fuzzy numbers in a path using α -cuts by proposing a linear least squares model to obtain membership functions for the considered additions. Then using a recently proposed distance function for comparison of fuzzy numbers, we propose a new approach to solve the fuzzy all pairs shortest path problem using a genetic algorithm. Examples are worked out to illustrate the applicability of the proposed model.

Keywords
α –cut, Distance function, Shortest path, Regression, Genetic Algorithm

1. Introduction
The problem of finding a shortest path from a specified source node to any other node is fundamental in graph theory and one that is of continuing interest [2,7]. Let $G = (V, E)$ be a graph, where V is the set of vertices (nodes) and E is the set of edges (arcs). A *path* between two nodes is an alternating sequence of vertices and edges beginning with a starting and ending with an ending node. The *distance* (cost) of a path is the sum of the weights (arc lengths) of the edges on the path.
Although in conventional graph theory, the weights of the edges in a shortest path problem (SPP) are assumed to be precise real numbers, for most practical applications, however, these parameters (i.e., costs, capacities, demands, time, etc.) are naturally imprecise. In such cases, an appropriate modeling approach may justifiably make use of fuzzy numbers, and so does the name *fuzzy shortest path problem* (FSPP) appear in the literature [2, 6, 10].
The work of Dubois and Prade [3] is one of the first on this subject and considers extensions of the classical Floyd and Ford–Moore–Bellman (FMB) algorithms. However, it was verified that the algorithm would compute the shortest path distance without identifying an existing path (see [4]) as it was outlined by Klein [5] with the fuzzy dominating set. Lin and Chern [6] defined the denomination of vital arcs as being those whose removal from the path resulted in an increase of the cost. Blue *et al*. [1] presented an algorithm which would find a cut value to limit the number of analyzed paths, and then applied a modified version of the *k*-shortest path (crisp) algorithm proposed by Eppstein [4]. Following the idea of finding a fuzzy set solution, Okada [11] introduced the concept of the degree of possibility of an arc being on a shortest path. Among the most recent work is the one by Nayeem and Pal [10] that proposes an algorithm based on the acceptance index introduced by Sengupta and Pal [13] and which gives a single fuzzy shortest path or a guideline for choosing a best fuzzy shortest path according to the decision-maker's viewpoint [8].
The remainder of the paper is organized as follows. In Section 2, basic concepts and definitions are given. Section 3 explains ways of computing $\alpha -$ cuts for fuzzy numbers. We present our fuzzy sum operator by use of a linear least squares model in Section 4. In Section 5, using a recently proposed distance function, we present a genetic algorithm

for finding fuzzy shortest path in a mixed fuzzy network. An example is worked out to illustrate the applicability of the proposed model in Section 6. We conclude in Section 7.

2. Definitions

The α-cut and strong α-cut for a fuzzy number $\tilde{a}$ are shown by $\tilde{a}_\alpha$ and $\tilde{a}_\alpha^+$, respectively, and for $\alpha \in [0,1]$ are defined to be:

$$\tilde{a}_\alpha = \{x \mid \mu_{\tilde{A}}(x) \geq \alpha, x \in X\},$$
$$\tilde{a}_\alpha^+ = \{x \mid \mu_{\tilde{A}}(x) > \alpha, x \in X\},$$

where X is the universal set.

3. Computing α-cuts for fuzzy numbers

For the LR fuzzy numbers with L and R invertible functions, the α -cuts are:

$$\alpha = L[\frac{m-x}{a}] \;\Rightarrow\; \frac{m-x}{a} = L^{-1}(\alpha) \;\Rightarrow\; \tilde{a}_\alpha^L = x = m - aL^{-1}(\alpha),$$
$$\alpha = R[\frac{x-m}{b}] \;\Rightarrow\; \frac{x-m}{b} = R^{-1}(\alpha) \;\Rightarrow\; \tilde{a}_\alpha^R = x = m + bR^{-1}(\alpha).$$

For specific L and R functions, the following cases are discussed.

3.1. α-cuts for trapezoidal fuzzy numbers

Let $\tilde{a} = (a_1, a_2, a_3, a_4)$ be a trapezoidal fuzzy number. An $\alpha -$ cut for $\tilde{a}$, $\tilde{a}_\alpha$, is computed as:

$$\alpha = \frac{x-a_1}{a_2-a_1} \Rightarrow \tilde{a}_\alpha^L = x = (a_2-a_1)\alpha + a_1,$$
$$\alpha = \frac{a_4-x}{a_4-a_3} \Rightarrow \tilde{a}_\alpha^R = x = a_4 - (a_4-a_3)\alpha,$$

$$\Rightarrow \tilde{a}_\alpha = \begin{cases} \tilde{a}_\alpha^L = (a_2-a_1)\alpha + a_1 \\ \\ \tilde{a}_\alpha^R = a_4 - (a_4-a_3)\alpha \end{cases}, \quad 0 \leq \alpha \leq 1, \tag{1}$$

where $\tilde{a}_\alpha = [\tilde{a}_\alpha^L, \tilde{a}_\alpha^R]$ is the corresponding α-cut.

3.2. α-cuts for normal fuzzy numbers

If $\tilde{a} = (m, \sigma)$ is a normal fuzzy number, then $\tilde{a}_\alpha$ is computed as:

$$\alpha = e^{-(\frac{m-x}{\sigma})^2} \Rightarrow \sqrt{-\ln(\alpha)} = \frac{m-x}{\sigma} \Rightarrow \tilde{a}_\alpha^L = x = m - \sigma\sqrt{-\ln\alpha}$$
$$\alpha = e^{-(\frac{x-m}{\sigma})^2} \Rightarrow \sqrt{-\ln(\alpha)} = \frac{x-m}{\sigma} \Rightarrow \tilde{a}_\alpha^R = x = m + \sigma\sqrt{-\ln\alpha}$$
$$\Rightarrow \tilde{a}_\alpha = \begin{cases} \tilde{a}_\alpha^L = m - \sigma\sqrt{-\ln\alpha} \\ \\ \tilde{a}_\alpha^R = m + \sigma\sqrt{-\ln\alpha} \end{cases}, \quad 0 < \alpha \leq 1. \tag{2}$$

4. Fuzzy approximate sum operators

Here, we use our recently proposed approach in [14] for summing various fuzzy numbers approximately using α-cuts. The approximation is based on fitting an appropriate model for the sum using α-cuts of the addition as the fitness data. Let us divide the α-interval [0,1] into n equal subintervals by letting $\alpha_0 = 0, \alpha_i = \alpha_{i-1} + \Delta\alpha_i$, $\Delta\alpha_i = \frac{1}{n}$, $i=1,\ldots,n$. This way, we have a set of $n+1$ equidistant points. For the normal fuzzy numbers $x \in (-\infty, +\infty)$, it is improper to assume α being equal to zero. Therefore, in this case we consider $\alpha \in (0,1]$, and thus use the nonzero α_i, $1 \le i \le n$.

4.1. $\alpha -$ Cut sum

Let $\tilde{a} = (a_1, a_2, a_3, a_4)$ and $\tilde{b} = (m, \sigma)$ be the trapezoidal and normal fuzzy numbers, respectively. Given $\alpha_i \in (0,1], 1 \le i \le n$, the α-cut sum of these fuzzy numbers using equations (1) and (2) is obtained as follows:

$$\tilde{c}_{\alpha_i} = \tilde{a}_{\alpha_i} + \tilde{b}_{\alpha_i} \Rightarrow \tilde{c}_{\alpha_i} = \begin{cases} \tilde{c}^L_\alpha = (\tilde{a}^L_\alpha + \tilde{b}^L_\alpha), \\ \\ \tilde{c}^R_\alpha = (\tilde{a}^R_\alpha + \tilde{b}^R_\alpha) \end{cases} \quad \forall\, i, \quad 1 \le i \le n,$$

where,

$$\tilde{c}^L_{\alpha_i} = [(a_2 - a_1)\alpha + a_1 + m - \sigma\sqrt{-\ln\alpha}],$$
$$\tilde{c}^R_{\alpha_i} = [a_4 - (a_4 - a_3)\alpha + m + \sigma\sqrt{-\ln\alpha}]. \tag{3}$$

Using equation (3), corresponding to α_i, $1 \le i \le n$, $2n$ points are obtained for $\tilde{c}$ (n points for the $\tilde{c}^L_{\alpha_i}$ and n points for the $\tilde{c}^R_{\alpha_i}$). Using these points, it is possible to approximate the sum of the two fuzzy numbers. An approximate membership function of the sum is computed by fitting an appropriate function using the α-cut points. Let $x_i = \tilde{c}^R_{\alpha_i}$ and $y_i = \mu(\tilde{c}^R_{\alpha_i})$, and using the n points (x_i, y_i), $1 \le i \le n$, consider the fitting model as $y = e^{-(\frac{x-\lambda}{\beta})^2}$. The unknown parameters λ and β appear nonlinearly. We linearize the model, by noting that for any $x_i > \lambda$ (as is the case here for the right hand model), β and λ are explicitly determined to be:

$$\beta = \frac{n\sum_i x_i \times \sqrt{-\ln y_i} - \sum_i \sqrt{-\ln y_i} \times \sum_i x_i}{-n\sum_i \ln y_i - \sum_i \sqrt{-\ln y_i} \times \sum_i \sqrt{-\ln y_i}}, \tag{4}$$

$$\lambda = \frac{-\sum_i \ln y_i \sum_i x_i - \sum_i \left(x_i \sqrt{-\ln y_i}\right) \times \sum_i \sqrt{-\ln y_i}}{-n\sum_i \ln y_i - \sum_i \sqrt{-\ln y_i} \times \sum_i \sqrt{-\ln y_i}}. \tag{5}$$

Now, similarly let $\bar{x}_i = \tilde{c}^L_{\alpha_i}$ and $\bar{y}_i = \mu(\tilde{c}^L_{\alpha_i})$, and consider the model $y = e^{-(\frac{\lambda' - x}{\beta'})^2}$. We then have

$$\beta' = \frac{-n\sum_i x_i \sqrt{-\ln y_i} + \sum_i x_i \times \sum_i \sqrt{-\ln y_i}}{-n\sum_i \ln y_i - \sum_i \sqrt{-\ln y_i} \times \sum_i \sqrt{-\ln y_i}}, \tag{6}$$

$$\lambda' = \frac{-\sum_i \ln y_i \times \sum_i x_i - \sum_i \left(x_i \times \sqrt{-\ln y_i}\right) \times \sum_i \sqrt{-\ln y_i}}{-n\sum_i \ln y_i - \sum_i \sqrt{-\ln y_i} \times \sum_i \sqrt{-\ln y_i}}. \tag{7}$$

Thus, the membership function is determined to be:

$$\mu_{\tilde{c}}(x) = \begin{cases} e^{-(\frac{\lambda' - x}{\beta'})^2} & x < \lambda' \\ 1 & \lambda' \le x \le \lambda \\ e^{-(\frac{x - \lambda}{\beta})^2} & x > \lambda, \end{cases} \tag{8}$$

with λ, β, λ' and β' as defined by (4), (5), (6), and (7), respectively.

5. An algorithm for fuzzy shortest path in a network

5.1. Distance between fuzzy numbers

Assume that $\tilde{a}$ and $\tilde{b}$ are two fuzzy numbers. We apply a fuzzy ranking method for fuzzy numbers. We have used this ranking method effectively in a recent work [8].Let us consider fuzzy *min* operations as follows:

$$\min(\tilde{a},\tilde{b}) = (\min(a_1,b_1), \min(a_2,b_2), \min(a_3,b_3), \min(a_4,b_4)). \tag{9}$$

It is evident that, for non-comparable fuzzy numbers $\tilde{a}$ and $\tilde{b}$, the fuzzy *min* operation results in a fuzzy number different from both of them. For example, for $\tilde{a} = (5,10,13,19)$ and $\tilde{b} = (6,9,15,20)$, we get from (9) a fuzzy $MV = \min(\tilde{a},\tilde{b}) = (5,9,13,19)$, which is different from both $\tilde{a}$ and $\tilde{b}$. To alleviate this drawback, we use a method based on the distance between fuzzy numbers. We use the distance function introduced in [12]. The main advantages of this distance function are the generality of its usage on various fuzzy numbers, and its reliability in distinguishing unequal fuzzy numbers. Indeed, the usage of the distance function below worked out to be quite appropriate for our approach.

The $D_{p,q}$ -distance, indexed by parameters $1 < p < \infty$ and $0 < q < 1$, between two fuzzy numbers $\tilde{a}$ and $\tilde{b}$ is a nonnegative function given by:

$$D_{p,q}(\tilde{a},\tilde{b}) = \begin{cases} \left[(1-q)\int_0^1 \left| a_\alpha^- - b_\alpha^- \right|^p d\alpha + q\int_0^1 \left| a_\alpha^+ - b_\alpha^+ \right|^p d\alpha \right]^{\frac{1}{p}}, & p<\infty, \\ (1-q) \sup\limits_{0<\alpha\le 1} \left(\left| a_\alpha^- - b_\alpha^- \right| \right) + q \inf\limits_{0<\alpha\le 1} \left(\left| a_\alpha^+ - b_\alpha^+ \right| \right), & p=\infty. \end{cases} \tag{10}$$

For two fuzzy numbers $\tilde{a}$ and $\tilde{b}$ with corresponding α_i -cuts, the $D_{p,q}$ distance is approximately proportional to :

$$D_{p,q}(\tilde{a},\tilde{b}) = \left[(1-q)\sum_{i=1}^{n} \left| a_{\alpha_i}^- - b_{\alpha_i}^- \right|^p + q\sum_{i=1}^{n} \left| a_{\alpha_i}^+ - b_{\alpha_i}^+ \right|^p \right]^{\frac{1}{p}}. \tag{11}$$

If $q = \frac{1}{2}, p = 2$, then the above equation turns into:

$$D_{2,\frac{1}{2}}(\tilde{a},\tilde{b}) = \sqrt{\frac{1}{2}\sum_{i=1}^{n} (a_{\alpha_i}^- - b_{\alpha_i}^-)^2 + \frac{1}{2}\sum_{i=1}^{n} (a_{\alpha_i}^+ - b_{\alpha_i}^+)^2}\ . \tag{12}$$

To compare two fuzzy arc lengths $\tilde{a}$ and $\tilde{b}$ with α_i -cuts as their approximations, since they are supposed to represent positive values, we compare them with $MV = (0,0,....,0)$. In fact, we use formula (12) to compute $D_{2,\frac{1}{2}}(\tilde{a},MV)$ and $D_{2,\frac{1}{2}}(\tilde{b},MV)$ and then use these values for comparison of the two numbers.

5.2 The Genetic Algorithm

5.2.1 Genetic Algorithm (GA) for solving shortest path problem with fuzzy arc lengths

Here, we describe the encoding scheme used by the GAs. Genetic operators specific to this encoding scheme are also defined. These include the initialization, crossover and mutation operators.

5.2.2 GA Encoding Scheme and Population Initialization

How to encode a path in a graph is critical for developing a genetic algorithm to this problem. This is not as easy as traveling salesmen problem to find out a natural representation. Special difficulties arise from (a) a path contains variable number of nodes and the maximal number is $n - 1$ for an n node graph, and (b) a random sequence of edges usually does not correspond to a path. To overcome such difficulties, we adopted an indirect approach: encode some guiding information for constructing a path, but a path itself, in a chromosome. The path is generated by the sequential node appending procedure beginning with the specified node 1 and terminating at the specified node n. A vector p is used to keep the trace of the nodes in the path.

Algorithm 1 (generating initial population)

(1) Find the vicinity matrix A of directed network $G = (N, A)$, determine *pop-size* and set $k = 1$.

(2) Set $i = 1$, $l = 1$ and $p(l)$=1.

(3) Select a member of $a^1(i)$, where $a^1(i) = \{ j \mid (i,j) \in A,\ a_{ij} = 1 \}$, and call it j Let $l = l + 1$ and $p(l)$=j.

(4) If $j \neq n$ then let $i=j$ and go to (3).
(5) Find produced path using the labels in the labeling vector *p*. Let $k = k + 1$.
(6) If $k \leq \textit{pop-size}$ then go to (2) else stop.

5.2.3 Crossover operator

Crossover combines information from two parents such that the two children have a "resemblance" to each parent. Standard crossovers such as one-point, two-point, and uniform are used in GA models. Two paths, called parents, are randomly selected from the population. Then, we select one or two common members (genes) and replace different sections from parents. Hence, two new children (chromosomes) are generated. It is obvious that the generated paths are feasible.

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5.2.4 Mutation operator

The traditional mutation operator mutates the gene's value randomly according to a small probability of mutation; thus, it is merely a random walk and does not guarantee a positive direction toward the optimal solution. The proposed heuristic mutation remedies this deficiency. In this scheme, the number of paths or selected chromosomes, *q*, for the mutation operator is determined using a mutation operator's rate or probability(p_m); the number of paths *q* is computed to be the product of population size and p_m, and *q* different random numbers are generated between 1 and population size to be used as number of selected paths. Then, a number, say *r*, between 1 and path length is selected randomly. The components 1 to *r*-1 is kept unchanged, and components *r* to path length are removed, replacing them with new nodes obtained by using Algorithm 1.

5.2.5 Evaluation and selection strategy

Considering population chromosome, a path for each chromosome is determined and the path length is considered as the chromosome value. Then, for comparison of chromosome values, we use (12) for finding distances. The new algorithm for finding shortest path follows here.

Algorithm 2 (finding shortest path)

Step 1. Find the possible paths from source vertex *s* to destination vertex *d* from among produced population in each repetition and compute the corresponding path lengths L_i, $i = 1,2,...,m$, for the possible *m* paths.

Step 2. Find the fuzzy shortest length $\tilde{L}^{\min}$ by the following steps:

Step 2-1. Set $\tilde{L}_{\min} = \tilde{L}_1$.

Step 2-2. for $i = 2,...,m$ do

Step 2 -3. Calculate $MV = Min(\tilde{L}_{\min}, \tilde{L}_i)$ by Eq. (9).

Step 2-4. Find the distance $D_{p,q}$ of MV from $\tilde{L}_{\min}$ and $\tilde{L}_i$ using Eq. (12):

$$D_1 = D_{p,q}(MV, \tilde{L}_{\min})$$

$$D_2 = D_{p,q}(MV, \tilde{L}_i)$$

$$\tilde{L}_{\min} = \arg\min(D_1, D_2).$$

6. Numerical Illustrations

Example 1:We consider a larger network having mixed arc lengths (a combination of normal and trapezoidal fuzzy numbers) and use our genetic algorithm to find the shortest paths. The arc lengths are specified in Table 1.

Table 1. The arc lengths.

arc	fuzzy number	arc	fuzzy number	arc	fuzzy number	arc	fuzzy number	arc	fuzzy number
(1,2)	(12,13,15,17)	(1,3)	(40,11)	(1,4)	(8,10,12,13)	(12,15)	(12,14,15,16)	(13,15)	(37,12)
(1,5)	(7,8,9,10)	(2,6)	(35,10)	(2,7)	(6,11,11,13)	(14,21)	(11,12,13,14)	(15,18)	(8,9,11,13)
(3,8)	(40,11)	(4,7)	(17,20,22,24)	(4,11)	(6,10,13,14)	(16,20)	(38,12)	(17,20)	(7,10,11,12)
(5,8)	(29,9)	(5,11)	(7,10,13,14)	(5,12)	(10,13,15,17)	(18,21)	(15,17,18,19)	(18,22)	(16,5)
(6,9)	(6,8,10,11)	(6,10)	(35,11)	(7,10)	(9,10,12,13)	(19,22)	(15,16,17,19)	(20,23)	(13,14,16,17)
(7,11)	(6,7,8,9)	(8,12)	(5,8,9,10)	(8,13)	(20,5)	(13,19)	(17,18,19,20)	(18,23)	(15,5)
(9,16)	(6,7,9,10)	(10,16)	(40,13)	(10,17)	(15,19,20,21)	(15,19)	(25,7)	(21,23)	(12,15,17,18)
(11,14)	(8,9,11,13)	(11,17)	(28,9)	(12,14)	(13,14,16,18)	(17,21)	(6,7,8,10)	(22,23)	(20,5)

Using the distance function $D_{p,q}$ (for q=1/2 and p=2), the shortest path from the source node 1 to the destination node 23 is determined to be: $1 \to 5 \to 12 \to 15 \to 18 \to 23$.
Figure 1 shows the convergence curve for Example 1.

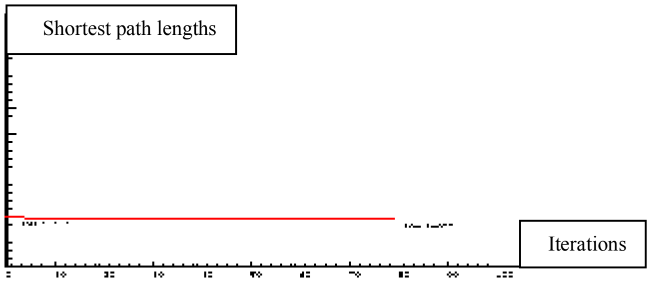

Figure 1. Convergence curve for Example 1.

By addition of various fuzzy numbers on the corresponding path, the membership function is obtained by (4)-(7) as follows:

$$\mu_{\tilde{c}}(x)=\begin{cases} e^{-\left(\frac{x-59}{9.154}\right)^2} & x<59 \\ 1 & 59<x<65 \\ e^{-\left(\frac{x-65}{8.559}\right)^2} & x>65. \end{cases}$$

7. Conclusions

We considered the problem of finding the all-pairs fuzzy shortest paths, with the lengths of arc given by fuzzy numbers. First, we proposed an order relation between fuzzy numbers. Then, we developed a fuzzy ranking method to avoid generating the set of non-dominated paths (or Pareto optimal paths) because the number of non-dominated paths derived from a large network can be too numerous, and it could be difficult for a decision maker to choose a preferable path. Then, we investigated the possibility of using genetic algorithm to solve fuzzy shortest path problems. The proposed approach was illustrated on a randomly generated problem having 23 nodes and 40 arcs.

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