

Optimal Control Policy of a Production and Inventory System for Deteriorating Items in Segmented Market

Kuldeep Chaudhary, Yogender Singh, P.C. Jha
Department of Operational Research
University of Delhi, Delhi, Delhi-110007, India

Abstract

Market segmentation is an essential element of marketing in industrialized countries. Goods can no longer be produced and sold without considering customer needs and recognized the heterogeneity of these needs. In this paper, we use market segmentation approach in single product inventory system with deteriorating items. The objective is to make use of optimal control theory to solve the production –inventory problem and develop an optimal production policy that maximize the total profit associated with inventory and production rate in segmented market. First, we consider a single production and inventory problem with multi-destination demand that vary from segment to segment. Further, we described a single source production –multi destination demand and inventory problem under the assumption that firm may choose independently the inventory directed to each segment. Both the problems are discussed and solved using Pontryagin Maximum principle.

Keywords

Production- Inventory System, Pontryagin Maximum Principle, Market Segmentation

1. Introduction

Market segmentation is an essential element of marketing in industrialized countries. Goods can no longer be produced and sold without considering customer's needs and recognizing the heterogeneity of these needs [1]. Earlier in this century, industrial development in various sectors of economy induced strategies of mass production and marketing. Those strategies were manufacturing oriented, focusing on reduction of production costs rather than satisfaction of customers. But as production processes become more flexible, and customer's affluence led to the diversification of demand, firms that identified the specific needs of groups of customers were able to develop the right offer for one or more sub –markets and thus obtained a competitive advantage.

Segmentation has emerged as a key planning tool and the foundation for effective strategy formulation. Nevertheless, market segmentation is not well known in mathematical inventory/production models. Only a few papers on inventory/production models deal with market segmentation (Duran et al. [2]; Chen and Li[3]).

Optimal control theory, a modern extension of the calculus of variations, is a mathematical optimization tool for deriving control policies. It has been used in inventory/ production [4-6] to derive the theoretical structure of optimal policies. Apart from inventory/ production, it has been successfully applied to many areas of operational research such as Finance [7,8], Economics[9-11], Marketing [12-15], Maintenance [16] and the Consumption of Natural Resources[17-19] etc.

The application of optimal control theory in inventory –production control analysis is possible due to its dynamic behavior. Continuous optimal control models provide a powerful tool for understanding the behavior of production/ inventory system where dynamic aspect plays an important role. Several papers have been written on the application of optimal control theory in Production-Inventory system with deteriorating items [20-23].

In this paper, we assume that firm has defined its target market in a segmented consumer population and that it develop a production –inventory plan to attack each segment with the objective of maximizing profit. In addition, we shed some light on the problem in the control of a single firm with a finite production capacity (producing a single item at a time) that serves as a supplier of a common product to multiple market segments. Segmented customers place demand continuously over time with rates that vary from segment to segment. In response to segmented customer demand, the firm must decide on how much inventory to stock and when to replenish this stock by producing. We apply optimal control theory to solve the problem and find the optimal production and inventory policy.

The rest of the paper is organized as follows. Following this introduction, all the notations and assumptions needed in the sequel is stated in Section 2. In section 3, we described the single source production and inventory problem with multi-destination demand that vary from segment to segment and develop the optimal control theory problem and its solution. In section 4 of this paper we introduce optimal control formulation of a single source production-multi destination demand and inventory problem and discuss its solution. A section 5 concludes the paper and provides some future research directions.

2. Notations and Assumptions

Here we begin the analysis by stating the model with as few notations as possible. Let us consider a manufacturing firm producing a single product in segmented market environment. We introduce the notation that is used in the development of the model: T : length of planning period, $P(t)$: Production rate at time t , $I(t)$: Inventory level at time t , $I_i(t)$: Inventory level at time t in i^{th} segment, $D_i(t)$: Demand rate at time t in i^{th} segment, $h(I(t))$: Holding cost rate, $h_i(I_i(t))$: Holding cost rate for i^{th} segment, c : The unit production cost rate, $\theta(t, I(t))$: Deterioration rate, $\theta_i(t, I_i(t))$: the deterioration rate in i^{th} segment at time t , $K(P(t))$: cost rate corresponding to the production rate, r_i : The revenue rate per unit sale in i^{th} segment, ρ : constant non-negative discount rate.

The model is based on the following assumptions: We assume that the time horizon is finite. The model is developed only for a single item in segmented market. The production, demand, and deterioration rates are function of time. The holding cost rate is function of inventory level & production cost rate depends on the production rate. The functions $h_i(I_i(t))$ (in case of single source $h(I(t))$) and $\theta_i(t, I_i(t))$ (in case of single source $\theta(t, I(t))$) are convex. All functions are assumed to be non negative, continuous and differentiable functions. This allows us to derive the most general and robust conclusions. Further, we will consider more specific cases for which we obtain some important results

3. Single Source Production and Inventory – Multi-Destination Demand Problem

Many manufacturing enterprises use a production –inventory system to manage fluctuations in consumers demand for the product. Such a system consists of a manufacturing plant and a finished goods warehouse to store those products which manufactured but not immediately sold. Here, we assume that once a product is made and put inventory into single warehouse, but demand for single product comes from each segment. Therefore, the inventory evolution in segmented market is described by the following differential equation:

$$\frac{d}{dt} I(t) = P(t) - \sum_{i=1}^n D_i(t) - \theta(t, I(t)) \quad (1)$$

So far, firm want to maximize the total Profit during planning period in segmented market. Therefore, the objective functional for all segments is defined as

$$Max \quad J = \int_0^T e^{-\rho t} \left(\sum_{i=1}^n r_i D_i(t) + c \left(\sum_{i=1}^n D_i(t) - P(t) \right) - K(P(t)) - h(I(t)) \right) dt \quad (2)$$

$P(t) \geq \sum_{i=1}^n D_i(t) + \theta(t, I(t))$

subject to the equation (1). This is the optimal control problem with one control variable(rate of production) with one state variable(inventory states). Since total demand occurs at rate $\sum_{i=1}^n D_i(t)$ and production occurs at controllable rate $P(t)$, it follows that $I(t)$ evolves according to the above state equation(1). The constraints $P(t) \geq \sum_{i=1}^n D_i(t) + \theta(t, I(t))$ and $I(0) = I_0 \geq 0$ ensure that shortage are not allowed.

Using the maximum principle [10], the necessary conditions for (P^*, I^*) to be an optimal solution of above problem are that there should exist a piecewise continuously differentiable function λ and piecewise continuous function μ , called the adjoint and Lagrange multiplier function, respectively such that

$$H(t, I^*, P^*, \lambda) \geq H(t, I^*, P, \lambda), \text{ for all } P(t) \geq \sum_{i=1}^n D_i(t) + \theta(t, I(t))$$

(3)

$$\frac{d}{dt} \lambda(t) = -\frac{\partial}{\partial I} L(t, I, P, \lambda, \mu) \quad (4)$$

$$I(0) = I_0, \lambda(T) = 0 \quad (5)$$

$$\frac{\partial}{\partial P} L(t, I, P, \lambda, \mu) = 0 \quad (6)$$

$$P(t) - \sum_{i=1}^n D_i(t) - \theta(t, I(t)) \geq 0, \quad \mu(t) \geq 0, \quad \mu(t) \left[P(t) - \sum_{i=1}^n D_i(t) - \theta(t, I(t)) \right] = 0 \quad (7)$$

Where, $H(t, I, P, \lambda)$ $L(t, I, P, \lambda, \mu)$ are Hamiltonian function and Lagrangian function respectively. In the present problem Hamiltonian function and Lagrangian function are defined as

$$H(t, I, P, \lambda) = \sum_{i=1}^n r_i D_i(t) + c \left(\sum_{i=1}^n D_i(t) - P(t) \right) - K(P(t)) - h(I(t)) + \lambda(t) \left(P(t) - \sum_{i=1}^n D_i(t) - \theta(t, I(t)) \right) \quad (8)$$

$$L(t, I, P, \lambda, \mu) = \sum_{i=1}^n r_i D_i(t) + c \left(\sum_{i=1}^n D_i(t) - P(t) \right) - K(P(t)) - h(I(t)) + (\lambda(t) + \mu(t)) \left(P(t) - \sum_{i=1}^n D_i(t) - \theta(t, I(t)) \right) \quad (9)$$

A simple interpretation of the Hamiltonian is that it represents the overall profit of the various policy decisions with both the immediate and the future effects taken into account and the value of $\lambda(t)$ at time t describes the future effect on profits upon making a small change in $I(t)$. Hence, the Hamiltonian H for all segments is strictly concave in $P(t)$ and according to the Mangasarian sufficiency theorem [4,10]; there exists a unique Production rate.

From equation (4) and (6), we have following equations respectively

$$\frac{d}{dt} \lambda(t) = \rho \lambda(t) - \left\{ -\frac{\partial h(I(t))}{\partial I} - (\lambda(t) + \mu(t)) \frac{\partial \theta(t, I(t))}{\partial I} \right\} \quad (10)$$

$$\lambda(t) + \mu(t) = c + \frac{d}{dP} K(P(t)) \quad (11)$$

Now, consider equation (7). Then for any t , we have either $P(t) - \sum_{i=1}^n D_i(t) - \theta(t, I(t)) = 0$ or $P(t) - \sum_{i=1}^n D_i(t) - \theta(t, I(t)) > 0$.

3.1. Case1:

Let S is a subset of planning period $[0, T]$, when $P(t) - \sum_{i=1}^n D_i(t) - \theta(t, I) = 0$. Then $\frac{d}{dt} I(t) = 0$ on S . in this case I^* is obviously constant on S and the optimal production rate is given by the following equation

$$P^*(t) = \sum_{i=1}^n D_i(t) - \theta(t, I^*(t)) \text{ for all } t \in S \quad (12)$$

By using equation (10) and (11), we have

$$\frac{d}{dt} \lambda(t) = \rho \lambda(t) - \left\{ -\frac{\partial h(I(t))}{\partial I} - \left(c + \frac{d}{dP} K(P(t)) \right) \frac{\partial \theta(t, I(t))}{\partial I} \right\} \quad (13)$$

After solving the above equation, we get an explicit form of the adjoint function $\lambda(t)$. From equation (10), we can obtain the value of Lagrange multiplier $\mu(t)$.

3.2. Case2:

$P(t) - \sum_{i=1}^n D_i(t) - \theta(t, I(t)) > 0$, for $t \in [0, T] \setminus S$. Then $\mu(t) = 0$ on $t \in [0, T] \setminus S$. In this case, the equation (10) and (11) becomes

$$\frac{d}{dt} \lambda(t) = \rho \lambda(t) - \left\{ -\frac{\partial h(I(t))}{\partial I} - \lambda(t) \frac{\partial \theta(t, I(t))}{\partial I} \right\} \quad (14)$$

$$\lambda(t) = c + \frac{d}{dP} K(P(t)) \quad (15)$$

Combining these equations with the state equation, we have the following second order differential equation:

$$\frac{d}{dt} P(t) - \frac{d^2}{dP^2} K(P) - \left[\rho + \frac{\partial \theta(t, I(t))}{\partial I} \right] \frac{d}{dP} K(P) = \rho c + c \frac{\partial \theta(t, I(t))}{\partial I} + \frac{\partial h(t, I(t))}{\partial I} \quad (16)$$

and $I(0) = I_0$, $c + \frac{d}{dP} K(P(T)) = 0$. For illustration purpose, let us assume the following forms the exogenous functions $K(P) = kP^2/2$, $h(t, I(t)) = hI(t)$, and $\theta(t, I(t)) = \theta_1 I(t)$, where k, h, θ_1 are positive constants.

For these functions the necessary conditions for (P^*, I^*) to be an optimal solution of problem (2) with equation (1) becomes

$$\frac{d^2}{dt^2} I(t) - \rho \frac{d}{dt} I(t) - (\rho + \theta_1) \theta_1 I(t) = \eta(t) \quad \text{with } I(0) = I_0, c + \frac{d}{dP} K(P(T)) = 0 \quad (17)$$

Where $\eta(t) = -\sum_{i=1}^n D_i + (\rho + \theta_1) \left(\sum_{i=1}^n D_i \right) + \frac{(c\rho + c\theta_1 + h)}{k}$. This problem is a two-point boundary value problem.

Proposition: The optimal solution of (P^*, I^*) to the problem is given by

$$I^*(t) = a_1 e^{m_1 t} + a_2 e^{m_2 t} + Q(t), \quad (18)$$

and the corresponding P^* is given by

$$P^*(t) = a_1 (m_1 + \theta_1) e^{m_1 t} + a_2 (m_2 + \theta_1) e^{m_2 t} + \frac{d}{dt} Q(t) + \theta_1 Q(t) + \sum_{i=1}^n D_i. \quad (19)$$

Where the constants a_1, a_2, m_1 , and m_2 are given in the proof attached in Appendix, and $Q(t)$ is a particular solution of the equation (17).

Proof: The solution of the two point boundary value problem(17) is given by standard method. It's characteristic equation $m^2 - \rho m - (\rho + \theta_1) \theta_1 = 0$, has two real roots of opposite sign, given by

$$m_1 = \frac{1}{2} \left(\rho - \sqrt{\rho^2 + 4(\rho + \theta_1) \theta_1} \right) < 0, \quad m_2 = \frac{1}{2} \left(\rho + \sqrt{\rho^2 + 4(\rho + \theta_1) \theta_1} \right) > 0, \text{ and therefore } I^*(t) \text{ is given}$$

by(18), where $Q(t)$ is the particular solution. Then initial and terminal condition used to determined the values of constant a_1 and a_2 as follows

$$a_1 + a_2 + Q(0) = I_0,$$

$$a_1 (m_1 + \theta_1) e^{m_1 T} + a_2 (m_2 + \theta_1) e^{m_2 T} + \left(\frac{c}{k} + \frac{d}{dt} Q(T) + \theta_1 Q(T) + \sum_{i=1}^n D_i(T) \right) = 0$$

By putting $b_1 = I_0 - Q(0)$ and $b_2 = -\left(\frac{c}{k} + \frac{d}{dt} Q(T) + \theta_1 Q(T) + \sum_{i=1}^n D_i(T) \right)$, we obtain the following system of two linear equations with two unknowns

$$\begin{aligned} a_1 + a_2 &= b_1, \\ a_1 (m_1 + \theta_1) e^{m_1 T} + a_2 (m_2 + \theta_1) e^{m_2 T} &= b_2 \end{aligned} \quad (20)$$

The value of P^* is deduced using the value of I^* and the state equation.

4. Single Source Production- Multi Destination Demand and Inventory Problem

We assume the single source production and multi destination demand-inventory system. Hence, the inventory evolution in each segmented is described by the following differential equation:

$$\frac{d}{dt} I_i(t) = \gamma_i P(t) - D_i(t) - \theta_i(t, I_i(t)) \quad \forall i \quad (21)$$

Here, we assume that $\gamma_i > 0$ and $\sum_{i=1}^n \gamma_i > 0$ with the conditions $I_i(0) = I_i^0$ and $\gamma_i P(t) \geq D_i(t) + \theta_i(t, I_i(t)) \quad \forall i$. We

called $\gamma_i > 0$ the segment production spectrum and $\gamma_i P(t)$ define the relative segment production rate towards segment i .

We develop a marketing-production model in which firm seeks to maximize its all profit by properly choosing production and market segmentation. Therefore, we defined the profit maximization objective function as follows:

$$\text{Max}_{\gamma_i P(t) \geq D_i(t) + \theta_i(t, I_i(t))} J = \int_0^T e^{-\rho t} \left(\sum_{i=1}^n r_i D_i(t) + c \sum_{i=1}^n (D_i(t) - \gamma_i P(t)) - \sum_{i=1}^n h_i(I_i(t)) - K(P(t)) \right) dt \quad (22)$$

subject to the equation (21). This is the optimal control problem (production rate) with one control variable with n state variable (stock of inventory in n segments).

To solve the optimal control problem expressed in equation (21) and (22), the following Hamiltonian and Lagrangian are defined as

$$H = \sum_{i=1}^n \left\{ r_i D_i(t) - h_i(I_i(t)) + c(D_i(t) - \gamma_i P(t)) \right\} - K(P(t)) + \sum_{i=1}^n \lambda_i(t) (\gamma_i P(t) - D_i(t) - \theta_i(t, I_i(t))) \quad (23)$$

$$L(t, I, P, \lambda, \mu) = \sum_{i=1}^n \left\{ r_i D_i(t) - h_i(I_i(t)) + c(D_i(t) - \gamma_i P(t)) \right\} - K(P(t)) + \sum_{i=1}^n (\lambda_i(t) + \mu_i(t)) (\gamma_i P(t) - D_i(t) - \theta_i(t, I_i(t))) \quad (24)$$

Equation (4), (6) and (21) yield

$$\frac{d}{dt} \lambda_i(t) = \rho \lambda_i(t) - \left\{ -\frac{\partial h_i(I_i(t))}{\partial I_i} - (\lambda_i(t) + \mu_i(t)) \frac{\partial \theta_i(t, I_i(t))}{\partial I_i} \right\} \quad \forall i \quad (25)$$

$$\sum_{i=1}^n (\lambda_i(t) + \mu_i(t)) \gamma_i = c \sum_{i=1}^n \gamma_i + \frac{d}{dP} K(P(t)) \quad (26)$$

In the next section of the paper, we consider only case when $\gamma_i P(t) - D_i(t) - \theta_i(t, I_i(t)) > 0 \quad \forall i$.

4.2.1. Case2:

$P(t) - \sum_{i=1}^n D_i(t) - \theta(t, I(t)) > 0$, for $t \in [0, T] \setminus S$. Then $\mu(t) = 0$ on $t \in [0, T] \setminus S$. In this case, the equation (25) and

(26) becomes

$$\frac{d}{dt} \lambda_i(t) = \rho \lambda_i(t) - \left\{ -\frac{\partial h_i(I_i(t))}{\partial I_i} - \lambda_i(t) \frac{\partial \theta_i(t, I_i(t))}{\partial I_i} \right\} \quad \forall i \quad (27)$$

$$\sum_{i=1}^n \lambda_i(t) \gamma_i = c \sum_{i=1}^n \gamma_i + \frac{d}{dP} K(P(t)) \quad (28)$$

Combining above equations with the state equation, we have the following second order differential equation:

$$\frac{d}{dt} P(t) \frac{d^2}{dP^2} K(P) - \sum_{i=1}^n \frac{1}{n} \left(\rho + \frac{\partial \theta_i(t, I_i(t))}{\partial I_i} \right) \frac{d}{dP} K(P) = \sum_{i=1}^n c \gamma_i \left(\rho + \frac{\partial \theta_i(t, I_i(t))}{\partial I_i} \right) + \sum_{i=1}^n \gamma_i \frac{\partial h_i(t, I_i(t))}{\partial I_i}$$

and $I_i(0) = I_i^0 \quad \forall i$, $c \sum_{i=1}^n \gamma_i + \frac{d}{dP} K(P(T)) = 0 \quad \therefore \lambda_i(T) = 0 \quad \forall i$. For illustration purpose, let us assume the following forms

the exogenous functions $K(P) = kP^2/2$, $h_i(t, I_i(t)) = h_i I_i(t)$, and $\theta_i(t, I_i(t)) = \theta_i I_i(t)$, where k, h_i, θ_i are positive constants.

For these functions the necessary conditions for (P^*, I_i^*) to be an optimal solution of problem (19) with equation (18) becomes

$$I_i''(t) + (\theta_i - A)I_i'(t) - A\theta_i I_i(t) = \eta_i(t) \quad \forall i \quad (29)$$

with $I_i(0) = I_i^0 \forall i$, $c \sum_{i=1}^n \gamma_i + \frac{d}{dP} K(P(T)) = 0$, because of $\lambda_i(T) = 0 \forall i$

Where $\eta_i(t) = -D_i A + \frac{\gamma_i}{k} \left[\sum_{i=1}^n \gamma_i (h_i + c(\rho + \theta_i)) \right] + D_i'(t)$ and $A = \sum_{i=1}^n \frac{(\rho + \theta_i)}{n}$. This problem is also a system of two-point boundary value problems.

The above system of two point boundary value problem (29) is solved by same method that we used in to solve (17).

5. Conclusion

In this paper, we have introduced market segmentation concept in the production inventory system and its optimal control formulation. We have used an optimal control theory approach to determine the optimal production rate policy that maximizes the total profit associated with inventory and production rate. The resulting analytical solution yield good insight on how production planning task can be carried out in segmented market environment. The present paper has been discussed under the assumption that the segmented demand is a function of time only. A natural extension to the analysis developed here is the consideration of segmented demand that is a general functional of time and amount of on-hand stock (inventory). Another interesting and quite difficult factor would be considering a multi-item approach in segmented market.

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