Proceedings of the 2011 International Conference on Industrial Engineering and Operations Management
Kuala Lumpur, Malaysia, January 22 – 24, 2011

Dynamically Stable Vertical Integration of Firms in a Supply Chain

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Abstract

Vertically integrating firms in a supply chain represents an effective way to enhance efficiency and overall profitability. However, vertical integrations could be unstable because some firms may gain sufficient advantages so that they would break away. For a cooperative scheme to be dynamically stable, it has to be time consistent. This paper derives a time-consistent cooperative scheme for a dynamic model of vertically integrated firms in a supply chain. A payoff distribution mechanism leading dynamically stable solution is derived. This is the first time that time-consistent vertical integration in supply chain is analyzed.

1. Introduction

A main tenet of modern supply chain management is that suppliers, manufacturers, wholesalers and retailers have a common interest in integrating and coordinating their activities to improve their overall performance. Vertically integrating firms in a supply chain represents an effective way to enhance efficiency and overall profitability. An essential condition for achieving this goal is the establishment of mutual trust among the chain members and the safe-guarding of the interests of various organizations. Laeequddin et al [1] developed a multi-level trust measurement instrument to measure supply chain members' trust and relationship. Richey et al [2] examined barriers to supply chain integration and firm's performance. Pfohl and Gomm [3] considered financial optimization in supply chains and Nilakantan [4] scrutinized the enhancement of chain performance with improved order-control tactics. Fabbe-Costes et al [5] investigated the role of logistic service in supply cahin integration. Work on supply chain collaboration and coordination can be found in [6-9]. Defee and Fugate [10] studied dynamic supply chain capabilities in today's evolving supply chain environment.

In spite of their purported benefits, vertical integrations could be unstable because after a certain time some firms may gain sufficient advantages so that they would do better by breaking away. A particularly stringent condition -- time consistency -- is required for a dynamically stable cooperative scheme. A cooperative scheme is time consistent if the agreed-upon optimality principle is maintained throughout the period of cooperation. Since all participants are guided by the same optimality principle at each instant of time, they do not possess incentives to deviate from the previously adopted optimal behavior. Examples of supply chains and marketing channels analysis involving a dynamic game framework can be found in [11-17]. Studies involving dynamically consistent solutions include [18-23]. This paper presents a dynamic model of vertically integrated firms in a supply chain. A time-consistent cooperative scheme is derived. This is the first time that time-consistent vertical integration in supply chain is analyzed.

2. A Dynamic Model of Supply Chain

Consider a $T-$stage $n-$firm supply chain which starts from raw materials supply, manufacturing, processing and so on to warehousing, distributing and retail.

2.1. Basic Settings

The profits of the firms are:

$$\sum_{\zeta=1}^{T} g_{\zeta}^{1}[x_{\zeta}^{1},x_{\zeta}^{2},u_{\zeta}^{1},u_{\zeta}^{2},x_{\zeta+1}^{1},x_{\zeta+1}^{2}]\left(\frac{1}{1+r}\right)^{\zeta-1} + S^{1}(x_{T}^{1},x_{T}^{2})\left(\frac{1}{1+r}\right)^{T},$$

$$\sum_{\zeta=1}^{T} g_\zeta^j[x_\zeta^{j-1},x_\zeta^j,x_\zeta^{j+1},u_\zeta^{j-1},u_\zeta^j,u_\zeta^{j+1},x_{\zeta+1}^{j-1},x_{\zeta+1}^j,x_{\zeta+1}^{j+1}]\left(\frac{1}{1+r}\right)^{\zeta-1}+S^j(x_T^{j-1},x_T^j,x_T^{j+1})\left(\frac{1}{1+r}\right)^T,$$

for $i\in\{2,\cdots,n-1\}$,

$$\sum_{\zeta=1}^{T} g_\zeta^n[x_\zeta^{n-1},x_\zeta^n,u_\zeta^{n-1},u_\zeta^n,x_{\zeta+1}^{n-1},x_{\zeta+1}^n]\left(\frac{1}{1+r}\right)^{\zeta-1}+S^n(x_T^{n-1},x_T^n)\left(\frac{1}{1+r}\right)^T, \tag{2.1}$$

where $u_k^i\in U^i\subset R^{m_i}$ is the control vector of firm $i\in\{1,2,\cdots,n\}\equiv\hat{N}$ at stage k, $x_k^i\in X^i$ is the state of firm i, and r is the discount rate. Note that there is a chain relationship which starts with firm 1 at the top of the chain to firm n at the bottom. $S^1(x_T^1,x_T^2)$, $S^j(x_T^{j-1},x_T^j,x_T^{j+1})$ and $S^n(x_T^{n-1},x_T^n)$ are terminal payments to the firms. Examples of the firms' state variables include capital stock, resource biomass, mineral deposits, level of technology and economic assets. Examples of the firms' controls include investment, production, resource extraction, R&D efforts, packing and distribution.

The state dynamics of the firms in the supply chain are:

$$x_{k+1}^1=f_k^1(x_k^1,x_k^2,u_k^1,u_k^2),\ x_1^1=x^{1*},$$
$$x_{k+1}^j=f_k^j(x_k^{j-1},x_k^j,x_k^{j+1},u_k^{j-1},u_k^j,u_k^{j+1}),\ x_1^j=x^{j*},\ \text{for } j\in\{2,\cdots,n-1\}$$
$$x_{k+1}^n=f_k^n(x_k^{n-1},x_k^n,u_k^{n-1},u_k^n),\ x_1^n=x^{n*};\ \text{for } k\in\{1,2,\cdots,T\}\equiv\kappa. \tag{2.2}$$

For notational convenience we use x_k to denote the vector $(x_k^1,x_k^2,\cdots,x_k^n)$.

2.2. Market Outcome

In this subsection, we characterize the noncooperative market outcome of the supply chain with firms' payoffs (2.1) and state dynamics (2.2). Let $\{\phi_k^i(x)$, for $k\in\kappa$ and $i\in N\}$ denote a set of strategies that provides a feedback Nash equilibrium solution (if it exists) to the supply chain (2.1)-(2.2), and

$$V^1(k,x)=\sum_{\zeta=k}^{T} g_\zeta^1[x_\zeta^1,x_\zeta^2,\phi_\zeta^1(x_\zeta),\phi_\zeta^2(x_\zeta),x_{\zeta+1}^1,x_{\zeta+1}^2]\left(\frac{1}{1+r}\right)^{\zeta-1}+S^1(x_T^1,x_T^2)\left(\frac{1}{1+r}\right)^T,$$

$$V^j(k,x)=\sum_{\zeta=k}^{T} g_\zeta^j[x_\zeta^{j-1},x_\zeta^i,x_\zeta^{j+1},\phi_\zeta^{j-1}(x_\zeta),\phi_\zeta^j(x_\zeta),\phi_\zeta^{j+1}(x_\zeta),x_{\zeta+1}^{j-1},x_{\zeta+1}^j,x_{\zeta+1}^{j+1}]\left(\frac{1}{1+r}\right)^{\zeta-1}$$
$$+S^j(x_T^{j-1},x_T^j,x_T^{j+1})\left(\frac{1}{1+r}\right)^T,\qquad \text{for } j\in\{2,\cdots,n-1\},$$

$$V^n(k,x)=\sum_{\zeta=k}^{T} g_\zeta^n[x_\zeta^{n-1},x_\zeta^n,\phi_\zeta^{n-1}(x_\zeta),\phi_\zeta^n(x_\zeta),x_{\zeta+1}^{n-1},x_{\zeta+1}^n]\left(\frac{1}{1+r}\right)^{\zeta-1}+S^n(x_T^{n-1},x_T^n)\left(\frac{1}{1+r}\right)^T,$$

denote the value functions indicating the game equilibrium payoff to the firms over the stages from k to T when $x_k\equiv(x_k^1,x_k^2,\cdots,x_k^n)=x\equiv(x^1,x^2,\cdots,x^n)$. A feedback Nash equilibrium of the game can be characterized as:

Theorem 2.1. A set of strategies $\{\phi_k^i(x)$, for $k\in\kappa$ and $i\in N\}$ provides a feedback Nash equilibrium solution to the game (1.1)-(1.2) if there exist functions $V^i(k,x)$, for $k\in K$ and $i\in N$, such that the following recursive relations are satisfied:

$$V^1(k,x)=$$
$$\max_{u_k^1}\left\{ g_k^1[x^1,x^2,u_k^1,\phi_k^2(x),\tilde{f}_k^{(1)1},\tilde{f}_k^{(1)2}],\left(\frac{1}{1+r}\right)^{k-1}+V^1(k+1,\tilde{f}_k^{(1)1},\tilde{f}_k^{(1)2})\right\},$$

$$V^j(k,x)=\max_{u_k^j}\left\{ g_k^j[x^{j-1},x^i,x^{j+1},\phi_k^{j-1}(x),u_k^j(x),\phi_k^{j+1}(x),\tilde{f}_k^{(j)j-1},\tilde{f}_k^{(j)j},f_k^{(j)j+1}]\left(\frac{1}{1+r}\right)^{k-1}\right.$$

$$+V^j(k+1,\ \widetilde{f}_k^{(j)j-1},\widetilde{f}_k^{(j)j},f_k^{(j)j+1})\quad\Big\},\ \text{for } j\in\{2,\cdots,n-1\},$$

$$V^n(k,x)=\max_{u_k^1}\Big\{\quad g_k^n[x^{n-1},x^n,\phi_k^{n-1}(x),u_k^n,\ \widetilde{f}_k^{(n)n-1},\widetilde{f}_k^{(n)n}],\ \left(\frac{1}{1+r}\right)^{k-1}+V^1(k+1,\ \widetilde{f}_k^{(n)n-1},\widetilde{f}_k^{(n)n})\quad\Big\},$$

$$V^1(T+1,x)=S^1(x^1,x^2)\left(\frac{1}{1+r}\right)^T,$$

$$V^j(T+1,x)=S^j(x^{j-1},x^j,x^{j+1})\left(\frac{1}{1+r}\right)^T,\quad \text{for } j\in\{2,\cdots,n-1\},\ \text{and}$$

$$V^n(T+1,x)=S^n(x^{n-1},x^n)\left(\frac{1}{1+r}\right)^T; \tag{2.3}$$

where

$\widetilde{f}_k^{(1)1}=f_k^1[x^1,x^2,u_k^1,\phi_k^2(x)]$,

$\widetilde{f}_k^{(1)2}=f_k^2[x_k^1,x_k^2,x_k^3,u_k^1,\phi_k^2(x),\phi_k^3(x)]$,

$\widetilde{f}_k^{(j)j-1}=f_k^{j-1}[x_k^{j-1},x_k^j,x_k^{j+1},\phi_k^{j-1}(x),u_k^j,\phi_k^{j+1}(x)]$, for $j\in\{2,\cdots,n-1\}$,

$\widetilde{f}_k^{(j)j}=f_k^j[x_k^{j-1},x_k^j,x_k^{j+1},\phi_k^{j-1}(x),u_k^j,\phi_k^{j+1}(x)]$, for $j\in\{2,\cdots,n-1\}$,

$\widetilde{f}_k^{(j)j+1}=f_k^{j+1}[x_k^{j-1},x_k^j,x_k^{j+1},\phi_k^{j-1}(x),u_k^j,\phi_k^{j+1}(x)]$, for $j\in\{2,\cdots,n-1\}$,

$\widetilde{f}_k^{(n)n-1}=f_k^{n-1}[x_k^{n-2},x_k^{n-1},x_k^n,\phi_k^{n-2}(x),\phi_k^{n-1}(x),u_k^n]$,

$\widetilde{f}_k^{(n)n}=f_k^n[x_k^{n-1},x_k^n,\phi_k^{n-1}(x),u_k^n]$.

Proof: (2.3) satisfy the optimality principle in discrete dynamic programming and the Nash equilibrium condition. Hence Theorem 2.1 follows. ■

3. Dynamic Cooperation

Now consider the case when the firms in the supply chain agree to cooperate and distribute the payoff among themselves according to an optimality principle. Two essential properties that a cooperative scheme has to satisfy are group optimality and individual rationality. Since firms in a supply chain are asymmetric, a reasonable optimality principle for gain distribution is to share the gain from cooperation proportional to the firms' relative sizes of noncooperative profits. We consider the following optimality principle

Principle PI.

Principle PI is an optimality principle which entails

(i) group optimality and

(ii) distribution of the gain from cooperation proportional to the firms' relative sizes of noncooperative profits. ■

3.1. Group Optimality

To maximize the supply chain's joint profit the firms have to solve the discrete-time dynamic programming problem of maximizing

$$\sum_{\zeta=1}^{T} g_\zeta^{*1}[x_\zeta^1,x_\zeta^2,u_\zeta^1,u_\zeta^2,x_{\zeta+1}^1,x_{\zeta+1}^2]\left(\frac{1}{1+r}\right)^{\zeta-1}$$
$$+\sum_{j=2}^{n-1}\left[\ \sum_{\zeta=1}^{T} g_\zeta^{*j}[x_\zeta^{j-1},x_\zeta^j,x_\zeta^{j+1},u_\zeta^{j-1},u_\zeta^j,u_\zeta^{j+1},x_{\zeta+1}^{j-1},x_{\zeta+1}^j,x_{\zeta+1}^{j+1}]\left(\frac{1}{1+r}\right)^{\zeta-1}\ \right]$$
$$+\sum_{\zeta=1}^{T} g_\zeta^{*n}[x_\zeta^{n-1},x_\zeta^n,u_\zeta^{n-1},u_\zeta^n,x_{\zeta+1}^{n-1},x_{\zeta+1}^n]\left(\frac{1}{1+r}\right)^{\zeta-1}+S^1(x_T^1,x_T^2)\left(\frac{1}{1+r}\right)^T$$

$$+\sum_{j=2}^{n-1} S^j(x_T^{j-1},x_T^j,x_T^{j+1})\left(\frac{1}{1+r}\right)^T + S^n(x_T^{n-1},x_T^n)\left(\frac{1}{1+r}\right)^T, \tag{3.1}$$

subject to (2.2).

As show in [3-5], various gains like those from financial optimization, improved order-control policies and logistic enhancement may be realized in the formation of a supply chain. If such gains exist in an integrated supply chain, the profits to firm j in an integration $g_\zeta^{j*}[x_\zeta^{j-1},x_\zeta^j,x_\zeta^{j+1},u_\zeta^{j-1},u_\zeta^j,u_\zeta^{j+1},x_{\zeta+1}^{j-1},x_{\zeta+1}^j,x_{\zeta+1}^{j+1}]$ is larger than $g_\zeta^j[x_\zeta^{j-1},x_\zeta^j,x_\zeta^{j+1},u_\zeta^{j-1},u_\zeta^j,u_\zeta^{j+1},x_{\zeta+1}^{j-1},x_{\zeta+1}^j,x_{\zeta+1}^{j+1}]$. Otherwise $g_\zeta^{j*}[\cdot]$ is identical to $g_\zeta^j[\cdot]$.

An optimal solution to the problem (3.1)-(2.2) can be characterized as:

Theorem 3.1. A set of strategies $\{\psi_k^i(x)$, for $k\in\kappa$ and $i\in N\}$ provides an optimal solution to the problem (3.1)-(2.2) if there exist functions $W(k,x)$, for $k\in K$, such that the following recursive relations are satisfied:

$$W(k,x)=\max_{u_k^1,u_k^2,\cdots,u_k^n}\Bigg\{ g_k^{*1}[x_k^1,x_k^2,u_k^1,u_k^2,f_k^1,f_k^2]\left(\frac{1}{1+r}\right)^{k-1}$$
$$+\sum_{j=2}^{n-1}\left[g_k^{*j}[x_k^{j-1},x_k^j,x_k^{j+1},u_k^{j-1},u_k^j,u_k^{j+1},f_k^{j-1},f_k^j,f_k^{j+1}]\left(\frac{1}{1+r}\right)^{k-1}\right]$$
$$+g_k^{*n}[x_k^{n-1},x_k^n,u_k^{n-1},u_k^n,f_k^{n-1},f_k^n]\left(\frac{1}{1+r}\right)^{k-1} + W[k+1,\ f_k^1,f_k^2,\cdots,f_k^n]\Bigg\}, \tag{3.2}$$

$$W(T+1,x)=S^1(x^1,x^2)\left(\frac{1}{1+r}\right)^T+\sum_{j=2}^{n-1} S^j(x^{j-1},x^j,x^{j+1})\left(\frac{1}{1+r}\right)^T+S^n(x^{n-1},x^n)\left(\frac{1}{1+r}\right)^T, \tag{3.3}$$

where $f_k^1,f_k^2,\cdots,f_k^n$ are short forms defined as follows:

$f_k^1=f_k^1(x_k^1,x_k^2,u_k^1,u_k^2)$, $f_k^j=f_k^j(x_k^{j-1},x_k^j,x_k^{j+1},u_k^{j-1},u_k^j,u_k^{j+1})$, $j\in\{2,\cdots,n-1\}$, and

$f_k^n=f_k^n(x_k^{n-1},x_k^n,u_k^{n-1},u_k^n)$.

Proof. (3.2) and (3.3) are direct optimality conditions from the method of dynamic programming. Hence Theorem 3.1 follows. ■

Substituting the optimal control $\{\psi_k^i(x)$, for $k\in\kappa$ and $i\in N\}$ into the state dynamics (2.2), one can obtain the dynamics of the cooperative trajectory which we denote by $\{x_k^*\}_{k=1}^T$.

3.2. Time-Consistent Solution and Payment Mechanism

To guarantee dynamical stability in a dynamic cooperation scheme, the solution has to satisfy the property of time consistency. In particular, the specific agreed-upon optimality principle must remain optimal at any stage of the game along the optimal state trajectory. Since at any stage of the game the agents are guided by the same optimality principles, and therefore do not have any ground for deviation from the previously adopted optimal behavior throughout the game. Let $\xi(k,x_k^*)=[\xi^1(k,x_k^*),\xi^2(k,x_k^*),\cdots,\xi^n(k,x_k^*)]$ denote the imputation vector.

Invoking optimality principle PI, we have to have

$$\xi^i(k,x_k^*)=\frac{V^i(k,x_k^*)}{\sum_{j=1}^n V^j(k,x_k^*)}W(k,x_k^*),\ \text{for } i\in N \text{ and } k\in\kappa. \tag{3.4}$$

Crucial to the analysis is the formulation of a payment mechanism so that the imputation in (3.4) can be realized. We use $B_k^i(x_k^*)$ to denote the payment that agent i will received at stage k under the cooperative agreement along the cooperative trajectory $\{x_k^*\}_{k=1}^T$. The payment scheme involving $B_k^i(x_k^*)$ constitutes a PDP in the sense that the imputation to agent i over the stages from k to T can be expressed as:

$$\xi^i(k,x_k^*) = B_k^i(x_k^*)\left(\frac{1}{1+r}\right)^{k-1} + \sum_{\zeta=k+1}^{T}\left\{ B_\zeta^i(x_\zeta^*)\left(\frac{1}{1+r}\right)^{\zeta-1}\right\}, \text{ for } i \in N \text{ and } k \in \kappa. \tag{3.5}$$

Using (3.5) one can obtain

$$\xi^i(k+1,x_{k+1}^*) = B_{k+1}^i(x_{k+1}^*)\left(\frac{1}{1+r}\right)^{k} + \sum_{\zeta=k+2}^{T}\left\{ B_\zeta^i(x_\zeta^*)\left(\frac{1}{1+r}\right)^{\zeta-1}\right\}. \tag{3.6}$$

Upon substituting (3.6) into (3.5) yields

$$\xi^i(k,x_k^*) = B_k^i(x_k^*)\left(\frac{1}{1+r}\right)^{k-1} + \xi^i[k+1, f_k(x_k^*,\psi_k(x_k^*))]. \tag{3.7}$$

Following the analysis of Yeung and Petrosyan [23] a Payoff Distribution Procedure (PDP) leading to imputations (3.4) can be obtained as:

Theorem 3.1.

A payment scheme with

$$B_k^j(x_k^*) = (1+r)^{k-1}\left[\ \xi^j(k,x_k^*) - \xi^j[k+1, f_1^*, f_2^*, \cdots, f_k^{n*}]\ \right], \text{ for } j \in N, \tag{3.8}$$

where $f_k^{1*} = f_k^1[x_k^1, x_k^2, \phi_k^1(x_k), \phi_k^2(x_k)]$,

$f_k^{j*} = f_k^j[x_k^{j-1}, x_k^j, x_k^{j+1}, \phi_k^{j-1}(x_k), \phi_k^j(x_k), \phi_k^{j+1}(x_k)]$, $j \in \{2,\cdots,n-1\}$, and

$f_k^{n*} = f_k^n[x_k^{n-1}, x_k^n, \phi_k^{n-1}(x_k), \phi_k^n(x_k)]$,

would lead to the realization of the imputation in (3.4).

Proof. From (3.7), one can readily obtain (3.8). ■

Substituting the state along the cooperative path $\left\{x_k^*\right\}_{k=1}^T$ and invoking (3.4), the PDP in Theorem 3.1 can be expressed as:

$$B_k^i(x_k^*) = (1+r)^{k-1}\left[\ \frac{V^i(k,x_k^*)}{\sum_{j=1}^{n} V^i(k,x_k^*)} W(k,x_k^*) - \frac{V^i(k+1,x_{k+1}^*)}{\sum_{j=1}^{n} V^i(k+1,x_{k+1}^*)} W(k+1,x_{k+1}^*)\ \right], \quad \text{for } i \in N. \tag{3.9}$$

4. An Illustration with Specific Functional Forms

Consider an illustration in which there are n firms in a supply chain. The profits of the firms are:

$$a^{(1)1}x_\zeta^1 + a^{(1)2}x_\zeta^2 - c^{(1)1}(u_\zeta^1)^2 + c^{(1)2}(u_\zeta^2)^2,$$

$$a^{(j)j-1}x_\zeta^{j-1} + a^{(j)j}x_\zeta^j + a^{(j)j+1}x_\zeta^{j-1} + c^{(j)j-1}(u_\zeta^{j-1})^2 - c^{(j)j}(u_\zeta^j)^2 + c^{(j)j+1}(u_\zeta^{j+1})^2,$$

for $j \in \{2,\cdots,n-1\}$,

$$a^{(n)n-1}x_\zeta^{n-1} + a^{(n)n}x_\zeta^n + c^{(n)n-1}(u_\zeta^{n-1})^2 - c^{(n)n}(u_\zeta^n)^2. \tag{4.1}$$

The state dynamics are

$$x_{\zeta+1}^1 = b^{(1)1}x_\zeta^1 + b^{(1)2}x_\zeta^2 + \omega^{(1)1}u_\zeta^1 + \omega^{(1)2}u_\zeta^2,\ x_1^1 = x^{1*},$$

$$x_{\zeta+1}^j = b^{(j)j-1}x_\zeta^{j-1} + b^{(j)j}x_\zeta^j + b^{(j)j+1}x_\zeta^{j+1} + \omega^{(j)j-1}u_k^{j-1} + \omega^{(j)j}u_k^j + \omega^{(j)j+1}u_k^{j+1},\ x_1^j = x^{j*},$$

for $j \in \{2,\cdots,n-1\}$,

$$x_{\zeta+1}^n = b^{(n)n-1}x_\zeta^{n-1} + b^{(n)n}x_\zeta^n + \omega^{(n)n-1}u_\zeta^{n-1} + \omega^{(n)n}u_\zeta^n,\ x_1^n = x^{n*}. \tag{4.2}$$

The supply chain game (4.1)-(4.2) is the discrete-time analogue of a sub-class of the class of differential games in Yeung [24]. Following his analysis the value functions can be obtained explicitly as:

$$V^i(k,x) = [\sum_{j=1}^{n} A_k^{(i)j} x_k^j + B_k^{(i)}] \left(\frac{1}{1+r}\right)^{k-1}, \text{ for } i \in N, \tag{4.3}$$

where $A_k^{(i)j}$ and $B_k^{(i)}$, for $i, j \in N$ and $k \in \kappa$, are constants.

Similarly, one can also obtain the explicitly the cooperative supply chain profit as:

$$W(k,x) = [\sum_{j=1}^{n} \hat{A}_k^j x_k^j + \hat{B}_k] \left(\frac{1}{1+r}\right)^{k-1}, \tag{4.4}$$

where $\hat{A}_k^j$ and $\hat{B}_k$, for $j \in N$ and $k \in \kappa$, are constants.

Invoking Theorem 3.1 a PDP leading to a time consistent solution under the optimality principle PI can be explicitly derived as

$$B_k^i(x_k^*) = [\sum_{j=1}^{n} A_k^{(i)j} x_k^{j*} + B_k^{(i)}][\sum_{j=1}^{n} \hat{A}_k^j x_k^{j*} + \hat{B}_k] \Big/ \sum_{h=1}^{n}[\sum_{j=1}^{n} A_k^{(h)j} x_k^{j*} + B_k^{(h)}]$$
$$-\left([\sum_{j=1}^{n} A_{k+1}^{(i)j} x_{k+1}^{j*} + B_{k+1}^{(i)}][\sum_{j=1}^{n} \hat{A}_{k+1}^j x_{k+1}^{j*} + \hat{B}_{k+1}] \Big/ \sum_{h=1}^{n}[\sum_{j=1}^{n} A_{k+1}^{(h)j} x_{k+1}^{j*} + B_{k+1}^{(h)}] \right)\left(\frac{1}{1+r}\right), \tag{4.5}$$

for $i \in N$ and $k \in \kappa$.

The above illustration shows that the solution can be obtained in closed-form. With given numerical values of the parameters $a^{(j)k}$, $c^{(j)k}$, $b^{(j)k}$ and $\omega^{(j)k}$, for $j,k \in N$, explicit results could be readily obtained from the formulae in (4.3), (4.4) and (4.5).

5. Conclusions and Recommendations

Vertically integrating firms in a supply chain represents an effective way to enhance efficiency and overall profitability. To secure a dynamically stable vertical integration a time consistent cooperative scheme has to be designed. This paper presents for the first time a time consistent vertical integration in supply chain. There are several directions that can be recommended for extension of this analysis. First more complex chain relationships can be incorporated. Second, stochastic elements can be introduced. Third, positive technological spillovers under integration can be incorporated into the state dynamics as:

$$x_{k+1}^j = f_k^{*j}(x_k^{j-1}, x_k^j, x_k^{j+1}, u_k^{j-1}, u_k^j, u_k^{j+1}), \text{ for } j \in N.$$

Since the analysis is formulated in a rather general framework, various functional forms pertinent to supply chains are applicable. Finally, further research along these lines should prove to be fruitful in the study of supply chain management.

Acknowledgements

Support by HK Research Grants Council CERG-HKBU202807 is gratefully acknowledged.

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