Threshold Function for the Optimal Stopping of Arithmetic Ornstein-Uhlenbeck Process

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Abstract

Mean reversion processes can be found in the core dynamics of numerous vital applications. In a mean reverting process, the value of the process tends to revert back to a long-run average value. In this paper, we study the optimal stopping problem of the widely used stochastic mean reverting process – the arithmetic Ornstein–Uhlenbeck process. We consider a single item that needs to be purchased within a given deadline, where its price process follows the arithmetic Ornstein–Uhlenbeck process. Methods to deal with this problem so far have focused on an infinite time horizon, which led to a constant threshold policy. We introduce an optimal policy for the finite time horizon case. The optimal policy is based on a recursive method to calculate a time-variant threshold function that represents the optimal stopping decision as a function of time. As a sub-routine of our method we develop explicit terms for the crossing time probability and the overshoot expectation of the mean reverting process. Then, we use these terms in the Bellman's equation of our model to construct the threshold function for the purchasing decision. Finally, we analyze the threshold function behavior with respect to its different parameters and derive meaningful properties.

1. Introduction

The optimal stopping time problem of the Ornstein-Uhlenbeck (*O-U*) process relates to a plethora of fields including applications in finance (see Bessembinder et al. 1995, Pindyck and Rubinfeld 1991, Schwartz 1997, Ekstrom et al. 2011), biology (Ricciardi and Sacerdote 1991), and physics (Tateno et al. 1995). In financial applications the process is typically used to model a price process whose dynamics exhibit some degree of mean reversion to a constant level. Examples of such models include the price process of commodities that is assumed to follow the geometric *O-U* process (see Bessembinder et al. 1995, Pindyck and Rubinfeld 1991, Schwartz 1997), and the spread process of two related assets under a pairs trading strategy that is assumed to follow the arithmetic *O-U* process (Ekstrom et al. 2011). In these kinds of models the problem of the optimal stopping time that determines the time to trigger an action, e.g., buy or sell the mean reverting asset, is crucial. Despite its main role, many aspects of the problem still lack a closed form solution. The general solution of this problem is determined in the form of a threshold policy that sets the price values across time that bounds the action region (Van Moerbeke 1976). For the case of infinite horizon, a constant threshold value can be calculated explicitly by solving a free boundary problem (Peskir and Shiryaev 2006). However, for the case of finite horizon no explicit analytic solution for calculating the optimal policy is known, but only numeric methods that are based on binomial and trinomial trees (Nelson and Ramaswamy 1990, Hull and White 1994), Monte Carlo simulation (Longstaff and Schwartz 2001), and on solving partial differential equations using finite difference method (Schwartz 1977). Other related sub-problems of the optimal stopping are the crossing time probability and the overshoot expectation of the *O-U* process with respect to some threshold level. These sub-problems present major challenges as well, and do not have explicit analytic solutions except for a few special cases. For the crossing time probability there are three main approximation methods: the eigenvalue expansion based method, the integral representation, and the 3-dimensional Bessel Bridge (Alili et al. 2005). All of these methods are limited to apply for a constant threshold only. In the discrete form of the *O-U* process (also known as the first order autoregressive, *AR(1)*, process), the overshoot expectation over a threshold does not have a known closed form solution as well, and may be derived using Monte Carlo simulation or by approximating the process to an autoregressive process with exponential stochastic terms (Jacobsen and Jensen 2007).

This work introduces a new approach to determine the optimal policy for purchasing a mean reverting asset whose price process follows the arithmetic *O-U* process when there is a finite planning horizon. The optimal policy is determined by a simple threshold function that divides the price region into a continuation region and a stopping region in which the asset is purchased. The threshold calculation method is based on solving the value function of the model according to the Bellman equation when decomposing the stochastic terms of the price process into a simplified form of a standard multivariate normal variable. Under this setting the crossing time probability and overshoot expectation of the process are derived and the threshold can be calculated recursively. This procedure outputs a simple threshold function, which is increasing towards the end of the planning horizon. To the best of our knowledge, a threshold function for the arithmetic *O-U* model has yet to be established.

The rest of the paper is organized as follows: We describe the model in section 2. We introduce the method for calculating the threshold function in section 3. Lastly, we analyze the threshold function properties in section 4.

2. Model

We consider a finite time horizon with length T. A single item has to be purchased by the end of the time horizon. Our goal is to minimize the expected purchase cost of the item. The item's price, denoted by x_t, fluctuates according to an arithmetic *Ornstein-Uhlenbeck* (*O-U*) process:

$$dx_t = K(\theta - x_t)dt + \sigma \cdot dW_t, \tag{1}$$

where θ is the long run price average, K is the reverting rate, dt is the time interval length, and σ is the degree of volatility. The stochastic term of the process, denoted by W_t, is modeled in the form of a Wiener process. Note that in this general model we allow the price process to reach a negative level, as it may represent the spread of two related assets in a pairs trading model (Ekstrom et al. 2011).

Let π denote a possible purchasing policy, and let $V_\pi(x_t, t)$ denote the expected purchasing cost under policy π. We will define π^* as the optimal policy that minimizes the expected purchasing cost. That is, $\pi^* = \arg\min V_\pi(x_t, t)$, and $V^*(x_t, t) = V_{\pi^*}(x_t, t)$. According to Van Moerbeke (1976), it is well known that the optimal policy is in the form of an increasing threshold function, $b(t)$. Hence, the optimal stopping problem can be reduced to finding the threshold function. However, in the case of a finite time horizon, no closed solution for the threshold function has been found.

3. The Threshold Policy

The following section characterizes the structure of the optimal policy. The optimal policy is based on a threshold function over time that states the following purchasing policy: buy if the item's price drops below the threshold, otherwise wait. A natural way to calculate the threshold function is by discretizing the price process and applying Bellman's equation: We get that at the end of the time horizon $V^*(x_T, T) = x_T$, as at time T the decision maker has no choice other than purchasing the item at price x_T. In order to simplify notation we define in the discrete form: $N = T/dt$, $n = t/dt$ and $t_i = i \cdot dt$ for $0 \le i \le N$, and let $x_0, x_{t_1}, x_{t_2}, \dots, x_{t_N}$ denote the discretized item price series. When modeling the process in the discrete form, it is important to note that the constraint $|1 - dtK| < 1$ must be added in order to keep the stationary property of the process.

According to the Bellman equation, we get that for $0 \le t_n < t_N$:

$$V^*(x_{t_n}, t_n) = \min\left\{E\left[V^*(x_{t_{n+1}}, t_{n+1})|x_{t_n}\right], x_{t_n}\right\}, \tag{2}$$

and the threshold value at time t_n is set to the value $b(t_n)$ that satisfies:

$$V^*(b(t_n), t_n) = b(t_n). \tag{3}$$

At the end of the time horizon, if the item has not yet been purchased it is purchased at any price. Hence, we can set the threshold value at the end of the horizon to infinity: $b(t_N) = \infty$. In addition, it can be easily seen that in the last period before the end of the time horizon, t_{N-1}, eq. (3) is satisfied when the process is at its long run average level. Therefore, we get that at time t_{N-1} the threshold function equals: $b(t_{N-1}) = \theta$.

3.1 Crossing Time Probability and Overshooting Expectation

Let τ denote the first crossing time of the process x_t with its respective threshold function, $b(t)$:

$$\tau = \inf\{t \ge 0: x_t \le b(t)\}.$$

We define $P_t(x_0)$ as the probability that the first crossing time occurs at time t, when the current price is x_0:

$$P_t(x_0) = \Pr(\tau = t|x_0),$$

and the overshoot expectation, $E_t(x_0)$, as the expectation of the item's price, conditioned that the first crossing time occurs at time t:

$$E_t(x_0) = E(x_t|x_0, \tau = t).$$

Note that in the continuation region, where $x_0 > b(0)$, we get that:

$$V^*(x_0, 0) = P_{t_1}(x_0) \cdot E_{t_1}(x_0) + \left(1 - P_{t_1}(x_0)\right) \cdot E\left[V^*(x_{t_1}, t_1)|x_0, x_{t_1} > b(t_1)\right]$$
$$= \sum_{i=1}^{N} P_{t_i}(x_0) \cdot E_{t_i}(x_0),$$

or for a general t:

$$V^*\left(x_{t_n}, t_n\right) = \sum_{i=n+1}^{N} P_{t_i}(x_{t_n}) \cdot E_{t_i}(x_{t_n}). \tag{4}$$

3.2. Decomposition to Multivariate Normal Variables

According to eq. (1) $x_{t_n} = x_{t_{n-1}} + dx_{t_{n-1}}$. By using (1) recursively, this equation can be extended to represent the item's price at time t_n, based on a known starting price at time 0:

$$x_{t_n} = \theta + (x_0 - \theta)(1 - dt \cdot K)^n. \tag{5}$$
$$+\sigma \sum_{i=1}^{n} (1 - dt \cdot K)^{i-1}\, dW_{t_{n-i}}.$$

Therefore x_{t_n} distributes as a normal random variable:

$$x_{t_n} \backsim N\left(\mu_{x_{t_n}}, \sigma^2_{x_{t_n}}\right),$$

where its expectation, $\mu_{x_{t_n}}$, and its variance, $\sigma^2_{x_{t_n}}$, equal:

$$\mu_{x_{t_n}} = \theta + (x_0 - \theta)(1 - dt \cdot K)^n, \tag{6}$$

$$\sigma^2_{x_{t_n}} = Var\left[\sigma \sum_{i=1}^{n} (1 - dt \cdot K)^{i-1}\, dW_{t_{n-i}}\right] \tag{7}$$
$$= dt \cdot \sigma^2 \frac{1 - (1 - dt \cdot K)^{2n}}{1 - (1 - dt \cdot K)^2}.$$

Let $z_{t_n} = \frac{x_{t_n} - \mu_{x_{t_n}}}{\sigma_{x_t}}$ be the corresponding standardized normal random variable of x_{t_n}. That is, $z_{t_n} \backsim N(0,1)$. As x_{t_i}, $1 \leq t_i \leq t_n$ are correlated, we define the corresponding symmetric covariance matrix of z_{t_i}, $1 \leq t_i < t_n$ by Σ_{t_n}. That is, $\Sigma_{t_n} = cov\left(z_{t_i}, z_{t_j}\right)$ for $1 \leq t_i, t_j \leq t_n$, where for $t_i \leq t_j$:

$$cov(z_{t_i}, z_{t_j}) = \frac{dt \cdot \sigma^2 \cdot (1 - dt \cdot K)^{j-i}}{\sigma_{x_{t_i}} \cdot \sigma_{x_{t_j}}} \cdot \frac{1 - (1 - dt \cdot K)^{2i}}{1 - (1 - dt \cdot K)^2}. \tag{8}$$

Proposition 1.

$$P_{t_n}(x_0) = F\left(-\widehat{\boldsymbol{B}}_{t_n}(x_0); \Sigma_{t_n}\right) - F\left(-\boldsymbol{B}_{t_n}(x_0); \Sigma_{t_n}\right), \tag{9}$$

where

$$\boldsymbol{B}_{t_n}(x_0) = \left[\beta_{t_1}(x_0), \beta_{t_2}(x_0), \ldots, \beta_{t_n}(x_0)\right],$$
$$\widehat{\boldsymbol{B}}_{t_n}(x_0) = \left[\hat{\beta}_{t_1}(x_0), \hat{\beta}_{t_2}(x_0), \ldots, \hat{\beta}_{t_n}(x_0)\right],$$

$$\beta_{t_i}(x_0) = \frac{b(t_i) - [\theta + (x_0 - \theta) \cdot (1 - dtK)^n]}{\sigma_{x_{t_n}}}, \tag{10}$$

$$\hat{\beta}_{t_i}(x_0) = \begin{cases} \beta_{t_i}(x_0), & 1 \leq i \leq n-1 \\ -\infty, & i = n \end{cases},$$

and $F(\boldsymbol{a}; \Sigma)$ is the cumulative distribution function of a standard multivariate normal variable with an $n \times n$ covariance matrix Σ, over the truncation vector $\boldsymbol{a} \in \mathbb{R}^n$.

Proof.

$$P_{t_n}(x_0) = \Pr\left(x_{t_n} \leq b(t_n), x_{t_{n-1}} > b(t_{n-1}), \ldots, x_{t_1} > b(t_1)|x_0\right)$$

$$= \Pr\left(z_{t_n} \le \beta_{t_n}(x_0), z_{t_{n-1}} > \beta_{t_{n-1}}(x_0), \dots, z_{t_1} > \beta_{t_1}(x_0) | x_0\right).$$

By the law of total probability we get:

$$\begin{aligned} P_{t_n}(x_0) &= \Pr\left(z_{t_{n-1}} \le -\beta_{t_{n-1}}(x_0), \dots, z_{t_1} \le -\beta_{t_1}(x_0) | x_0\right) \\ &\quad - \Pr\left(z_{t_n} \le -\beta_{t_n}(x_0), z_{t_{n-1}} \le -\beta_{t_{n-1}}(x_0), \dots, z_{t_1} \le -\beta_{t_1}(x_0) | x_0\right). \\ &= F\left(-\widehat{\boldsymbol{B}}_{t_n}(x_0); \Sigma_{t_n}\right) - F\left(-\boldsymbol{B}_{t_n}(x_0); \Sigma_{t_n}\right). \end{aligned}$$

■

Let

$$\boldsymbol{A}_{t_n}^{t_j}(x_0) = \left[\alpha_{t_1}^{t_j}(x_0), \dots, \alpha_{t_{j-1}}^{t_j}(x_0), \alpha_{t_{j+1}}^{t_j}(x_0), \dots, \alpha_{t_n}^{t_j}(x_0)\right],$$
$$\widehat{\boldsymbol{A}}_{t_n}^{t_j}(x_0) = \left[\hat{\alpha}_{t_1}^{t_j}(x_0), \dots, \hat{\alpha}_{t_{j-1}}^{t_j}(x_0), \hat{\alpha}_{t_{j+1}}^{t_j}(x_0), \dots, \hat{\alpha}_{t_n}^{t_j}(x_0)\right],$$

where

$$\alpha_{t_i}^{t_j}(x_0) = \left(\beta_{t_i}(x_0) - Cov\left(z_{t_i}, z_{t_j}\right) \cdot \beta_{t_j}(x_0)\right) / \sqrt{1 - Cov\left(z_{t_i}, z_{t_j}\right)^2}, 1 \le i, j \le n,\ i \ne j,$$

$$\hat{\alpha}_{t_i}^{t_j}(x_0) = \begin{cases} \alpha_{t_i}^{t_j}(x_0), & 1 \le i \le n-1, 1 \le j \le n, i \ne j \\ -\infty, & i = n \end{cases},$$

and let $\boldsymbol{M}_{t_n}^{t_j}$ be the $n-1 \times n-1$ first-order partial correlation matrix of z_{t_i}, for $1 \le i, j \le n, i \ne j$, when removing the controlling variable z_{t_j}. That is:

$$\boldsymbol{M}_{t_n}^{t_j} = \rho_{t_i, t_k . t_j},\ 1 \le i, j, k \le n,\ i, j \ne k,$$

where

$$\rho_{t_i, t_k . t_j} = \frac{cov\left(z_{t_i}, z_{t_k}\right) - cov\left(z_{t_i}, z_{t_j}\right) \cdot cov\left(z_{t_k}, z_{t_j}\right)}{\sqrt{1 - cov\left(z_{t_i}, z_{t_j}\right)^2} \cdot \sqrt{1 - cov\left(z_{t_k}, z_{t_j}\right)^2}}.$$

By using Tallis' equation to calculate the conditional expectation of an element from a standard multivariate normal variable (Tallis 1961), we derive the terms for calculating $E_{t_n}(x_0)$. First we use Tallis' equation to calculate the terms $E\left(x_{t_n} | x_0, \tau > t_n\right)$ and $E\left(x_{t_n} | x_0, \tau > t_{n-1}\right)$ and get:

$$E\left(z_{t_n} | x_0, \tau > t_n\right) = \frac{\sum_{j=1}^{n} cov\left(z_{t_n}, z_{t_j}\right) \cdot \phi\left(\beta_{t_j}(x_0)\right) \cdot F\left(\boldsymbol{A}_{t_n}^{t_j}(x_0); \boldsymbol{M}_{t_n}^{t_j}\right)}{F\left(\boldsymbol{B}_{t_n}(x_0); \Sigma_{t_n}\right)}, \tag{11}$$

$$E\left(z_{t_n} | x_0, \tau > t_{n-1}\right) = \frac{\sum_{j=1}^{n} cov\left(z_{t_n}, z_{t_j}\right) \cdot \phi\left(\hat{\beta}_{t_j}(x_0)\right) \cdot F\left(\widehat{\boldsymbol{A}}_{t_n}^{t_j}(x_0); \boldsymbol{M}_{t_n}^{t_j}\right)}{F\left(\widehat{\boldsymbol{B}}_{t_n}(x_0); \Sigma_{t_n}\right)}, \tag{12}$$

where $\phi(x)$ is the probability density function (PDF) of a standard normal variable.

Therefore,

$$\begin{aligned} &E\left(x_{t_n} | x_0, \tau > t_n\right) \\ &= \mu_{x_{t_n}} + \frac{\sigma_{x_{t_n}} \cdot \sum_{j=1}^{n} cov\left(z_{t_n}, z_{t_j}\right) \cdot \phi\left(\beta_{t_j}(x_0)\right) \cdot F\left(\boldsymbol{A}_{t_n}^{t_j}(x_0); \boldsymbol{M}_{t_n}^{t_j}\right)}{F\left(\boldsymbol{B}_{t_n}(x_0); \Sigma_{t_n}\right)}, \end{aligned} \tag{13}$$

$$\begin{aligned} &E\left(x_{t_n} | x_0, \tau > t_{n-1}\right) \\ &= \mu_{x_{t_n}} + \frac{\sigma_{x_{t_n}} \cdot \sum_{j=1}^{n} cov\left(z_{t_n}, z_{t_j}\right) \cdot \phi\left(\hat{\beta}_{t_j}(x_0)\right) \cdot F\left(\widehat{\boldsymbol{A}}_{t_n}^{t_j}(x_0); \boldsymbol{M}_{t_n}^{t_j}\right)}{F\left(\widehat{\boldsymbol{B}}_{t_n}(x_0); \Sigma_{t_n}\right)}. \end{aligned} \tag{14}$$

Then, by the law of total expectation, using eqs. (9, 13, 14) we get that:

$$E_{t_n}(x_0) = \frac{\left[E\left(x_{t_n} | x_0, \tau > t_{n-1}\right) - \left(1 - P_{t_n}(x_0)\right) \cdot E\left(x_{t_n} | x_0, \tau > t_n\right)\right]}{P_{t_n}(x_0)}. \tag{15}$$

3.3 Threshold Calculation

The threshold values can be calculated by satisfying eq. (3), where by eqs. $(9, 15)$ the value function can be derived. However, at time t_n eqs. $(9, 15)$ require the values of $b(t_i)$ for $n < i \le N$ as an input. Therefore, we obtain $b(t_n)$ by following a recursive procedure that starts with the calculation of $b(t_N) = \infty, b(t_{N-1}) = \theta$, and then proceeds to lower indexes of t sequentially. In each step the value of $b(t_i)$ that satisfies eq. (3) is derived by a simple search that can be designed for any precision desired. Figure 1 exemplifies the threshold function,

$b(t)$, for different parameters.

4. Threshold Properties

In this section we introduce some meaningful properties of the threshold function.

Lemma 1. θ is the intercept of the threshold function.

Proof. By the definition of the price process in eq. (1), the only term of θ that affects the process is the difference between θ and x_t. That is, only this difference affects the process and not the price level itself.■

Lemma 2. $b(t)$ is linear in σ.

Proof. According to Lemma 1, θ acts only as the intercept of the threshold. Hence, without loss of generality, we

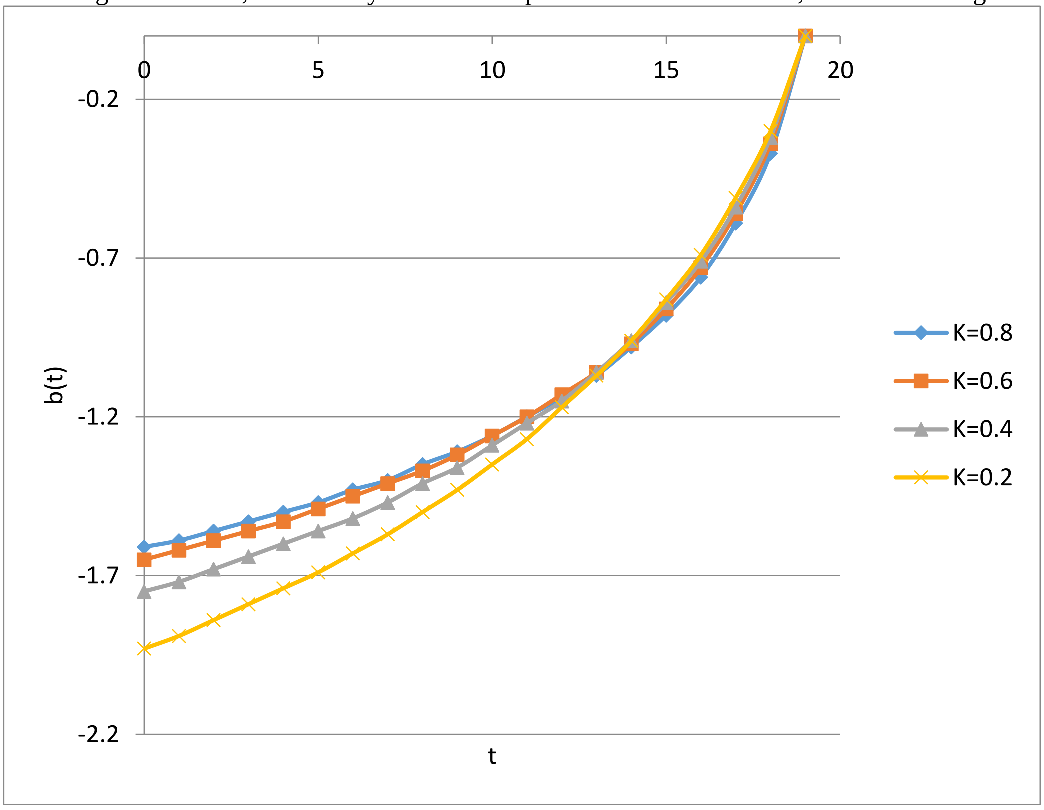

Figure 1: The threshold function for $\theta = 0, \sigma = 1, dt = 1,$ across $K = 0.2, 0.4, 0.6, 0.8$ for a time horizon of $T = 20$.

may assume that $\theta = 0$ for this proof. Let us assume that

Assumption 1: for $1 \le i \le N$, $b(t_i)$ is linear in σ.

Next, we prove that under Assumption 1, $b(t_0)$ is linear in σ as well. Let x'_t denote the price process where $\sigma = 1$, which follows the process $dx'_t = K(\theta - x'_t)dt + dW_t$. Respectively, let τ', z'_{t_i}, $b'(t_i)$, $\beta'_{t_i}(x'_0)$, $\boldsymbol{B}'_{t_i}(x'_0)$, $\widehat{\boldsymbol{B}}'_{t_i}(x'_0)$, Σ'_{t_i}, $P'_{t_i}(x'_0)$, $E'_{t_n}(x'_0)$, $V^{*\prime}(x'_0, t_0)$ denote the corresponding x'_t process-related terms of τ, z_{t_i}, $b(t_i)$, $\beta_{t_i}(x_0)$, $\boldsymbol{B}_{t_i}(x_0)$, $\widehat{\boldsymbol{B}}_{t_i}(x_0)$, Σ_{t_i}, $P_{t_i}(x_0)$, $E_{t_n}(x_0)$, $V^*(x_0, t_0)$. According to eqs. $(7,8)$, $cov\left(z_{t_i}, z_{t_j}\right)$, for $1 \le i,\ j \le n$ is independent in σ. Looking carefully at x_t, the general price process with a volatility degree, σ, in comparison to x'_t, we get that according to eq. (10) under Assumption 1,

$$\beta_{t_i}(\sigma \cdot x'_0) = \beta'_{t_i}(x'_0) \text{ for } 1 \le i \le N.$$

Therefore,

$$F\left(-\boldsymbol{B}_{t_n}(\sigma \cdot x'_0); \Sigma_{t_n}\right) = F\left(-\boldsymbol{B}'_{t_n}(x'_0); \Sigma'_{t_n}\right),$$

and

$$F\left(-\widehat{\boldsymbol{B}}_{t_n}(\sigma \cdot x'_0); \Sigma_{t_n}\right) = F\left(-\widehat{\boldsymbol{B}}'_{t_n}(x'_0); \Sigma'_{t_n}\right).$$

Hence, according to eq. (9) we get that

$$P_{t_n}(\sigma \cdot x'_0) = P_{t_n}{}'(x'_0)$$

under Assumption 1. Similarly, according to eqs. $(11,12)$ we get that under Assumption 1

$$E\left(z_{t_n} | \sigma \cdot x'_0, \tau > t_n\right) = E\left(z'_{t_n} | x'_0, \tau' > t_n\right),$$

and

$$E\left(z_{t_n} | \sigma \cdot x_0', \tau > t_{n-1}\right) = E\left(z'_{t_n} | x_0', \tau' > t_{n-1}\right).$$

Hence, according to eqs. $(6, 7, 13, 14)$, we get that under Assumption 1,

$$\frac{E\left(x_{t_n} | \sigma \cdot x_0', \tau > t_n\right)}{\sigma} = E\left(x'_{t_n} | x_0', \tau' > t_n\right),$$

and

$$\frac{E\left(x_{t_n} | \sigma \cdot x_0', \tau > t_{n-1}\right)}{\sigma} = E\left(x'_{t_n} | x_0', \tau' > t_{n-1}\right),$$

and therefore according to eq. (15),

$$E_{t_n}(\sigma \cdot x_0')/\sigma = E'_{t_n}(x_0').$$

Finally, we get that according to eq. (4), under Assumption 1,

$$V^*(\sigma \cdot x_0', t_0)/\sigma = V^{*\prime}(x_0', t_0).$$

By the definition of the threshold value,

$$V^{*\prime}(b'(t_0), t_0) = b'(t_0),$$

and therefore,

$$V^*(\sigma \cdot b'(t_0), t_0) = \sigma \cdot b'(t_0).$$

That is, the threshold values are linear in σ under Assumption 1. Moreover, at the end of the time horizon, $b(t_{N-1}) = 0$ for any σ, and trivially satisfies:

$$V^*(\sigma \cdot b'(t_{N-1}), t_{N-1}) = \sigma \cdot b'(t_{N-1}).$$

Therefore $b(t_{N-1})$ is in fact linear in σ, which makes the induction assumption valid.■

Figure 2 shows the linear behavior of the threshold function for different deadlines.

Lemma 3. For a constant ratio of $\frac{dt}{K}$, $b(t)$ is linear in $\sqrt{dt}$ and in $\frac{1}{\sqrt{K}}$.

Proof. According to eqs. $(6, 7, 8)$, the parameters: dt, K, σ appear only in two forms: $dt \cdot K$ and $dt \cdot \sigma^2$, or can be easily be manipulated to appear in the forms of: $dt \cdot K$ and $\frac{1}{K} \cdot \sigma^2$. Therefore, it is straight forward to apply the same induction procedure as in Lemma 2 to show that $b(t)$ is linear in $\sqrt{dt}$, and $\frac{1}{\sqrt{K}}$, under a constant value of $dt \cdot K$. ■

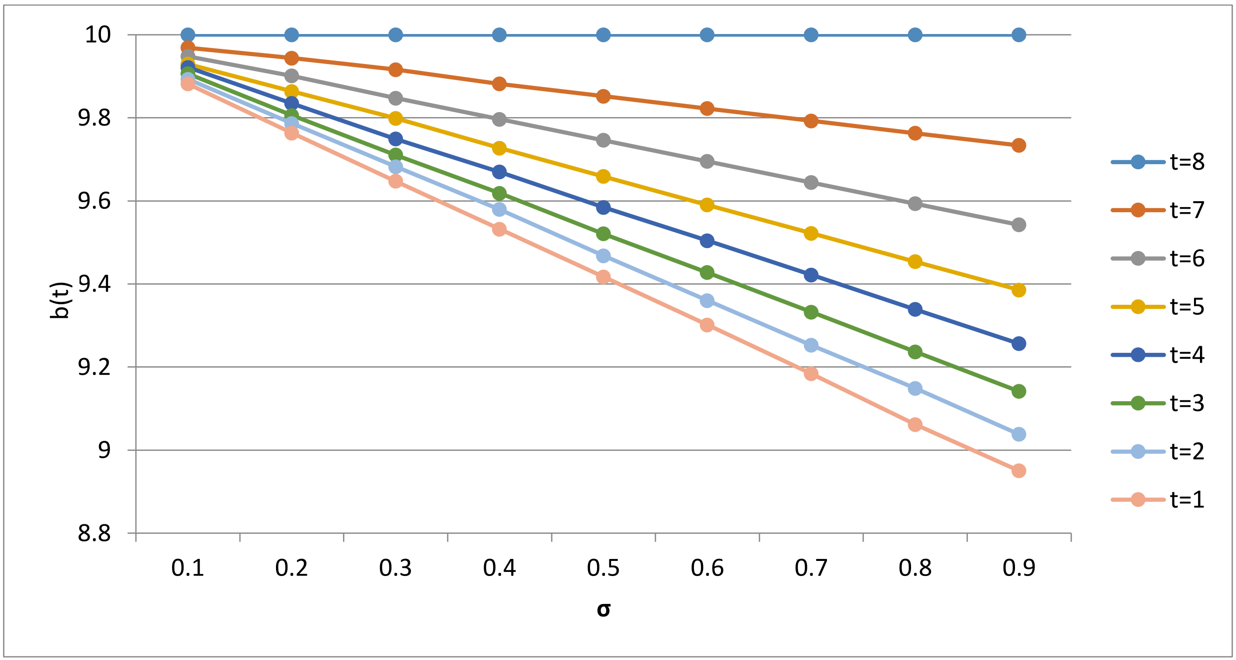

Figure 2: Threshold values as a function of σ, with $\theta = 10$, $dt = 1$, $K = 0.2$, $T = 8$ at different times: $t = 1, 2, 3, 4, 5, 6, 7, 8$.

5. Conclusion

In this paper, we introduced a simple analytical method to solve the optimal stopping problem of an arithmetic *O-U* process. The optimal policy is in the form of a simple threshold function, which indicates whether the process should be stopped. The method is based on finding explicit terms for the related crossing time probability and overshoot expectation by decomposing the terms to the form of multivariate normal variables. We show that the threshold function is linear in σ, and for a constant $dt \cdot K$ value it is linear in $\sqrt{dt}$, and $\frac{1}{\sqrt{K}}$. The technique developed in this research could provide practical tools in financial instruments and other *O-U* related areas such as biology and physics.

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Biography

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